# Monte Carlo Methods in Filtering

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# Robot Localization

- Where am I?


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- Want: Uncertain robot state at given timestep $\mathbf{x}_k$.


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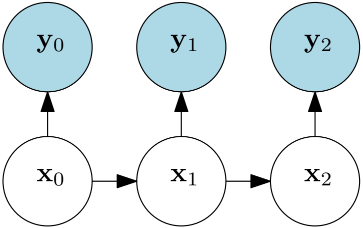
- Hidden Markov Model.



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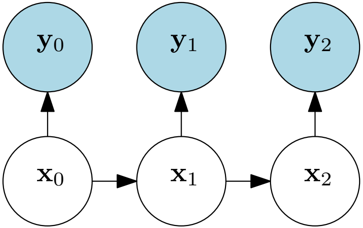


- Conditional measurement independence: Each measurement $\mathbf{y}_k$ is conditionally independent given the state it depends on,

$$p(\mathbf{y}_k|\mathbf{x}_{1:k}, \mathbf{y}_{1:k}) = p(\mathbf{y}_k|\mathbf{x}_k). \tag{1}$$

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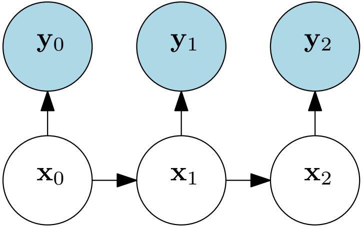


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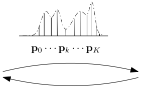
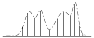
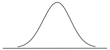
- Markov assumption: Each state is conditionally independent of previous measurements given the previous hidden state.

$$p(\mathbf{x}_k|\mathbf{x}_{1:k-1}, \mathbf{y}_{1:k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1}). \tag{2}$$

# Non-Exhaustive Taxonomy of Estimation Methods

| | Non-Parametric | Parametric |
|---|---|---|
| Smoothed $p(\mathbf{x}_k\|\mathbf{y}_{1:\boxed{K}})$ | Particle Smoother  $\mathbf{p}_0 \cdots \mathbf{p}_k \cdots \mathbf{p}_K$ <br> $\mathbf{x}_0 \; \cdots \; \mathbf{x}_k \; \cdots \; \mathbf{x}_N$ <br> Normalizing Flow <br> iSAM <br> Nested Sampling on Factor Graphs | Batch  $\mathbf{x}_0 \; \cdots \; \mathbf{x}_k \; \cdots \; \mathbf{x}_N$ <br> Sliding Window  $\mathbf{x}_0 \; \mathbf{x}_1 \; \cdots \; \mathbf{x}_{k-2} \; \mathbf{x}_{k+2} \; \cdots \; \mathbf{x}_N$ <br> RTS Smoother  $\mathbf{x}_0 \; \cdots \; \mathbf{x}_k \; \cdots \; \mathbf{x}_N$ |
| Filtering $p(\mathbf{x}_k\|\mathbf{y}_{1:\boxed{k}})$ | Particle Filter  $\mathbf{p}_0 \cdots \mathbf{p}_k \cdots \mathbf{p}_K$ | Gaussian Filter  $\left\{\begin{array}{l}\text{Extended (linearization)} \\ \text{Sigma-point (numerical integration)}\end{array}\right.$ |

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$$p(\mathbf{x}_k|\mathbf{y}_{1:k}) = \frac{1}{\eta} p(\mathbf{y}_k|\mathbf{x}_k) p(\mathbf{x}_k|\mathbf{y}_{1:k-1}). \tag{4}$$

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- Reverse marginalization gives *Chapman-Kolmogorov* equation

$$\begin{aligned} p(\mathbf{x}_k|\mathbf{y}_{1:k-1}) &= \int p(\mathbf{x}_k, \mathbf{x}_{k-1}|\mathbf{y}_{1:k-1})\mathrm{d}\mathbf{x}_{k-1} & (5) \\ &= \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-1})\mathrm{d}\mathbf{x}_{k-1}. & (6) \end{aligned}$$

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- Choice of how to parametrize state belief.
  - Parametric: Gaussian, multimodal.
  - Non-parametric: Particles.

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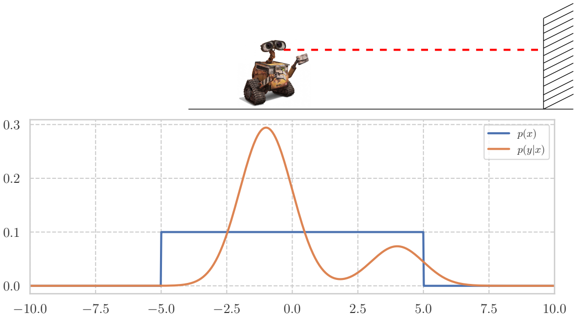

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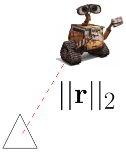


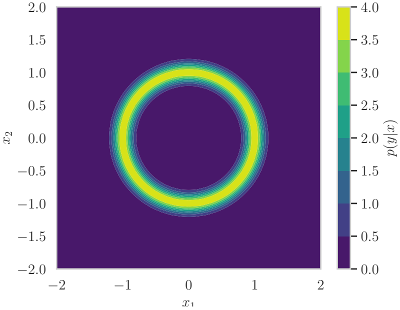

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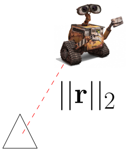
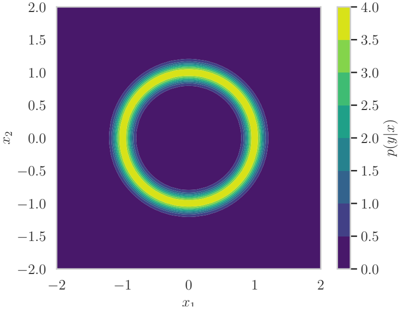
  - Strongly nonlinear models, ambiguous data associations, loop closures



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- PDF expressed as

$$\boxed{p(\mathbf{x}) \approx \sum_{i=1}^{N} w_i \delta(\mathbf{x} - \mathbf{x}_i), \quad \sum_{i=1}^{N} w_i = 1.} \tag{13}$$

# General Sampling Methods: Importance Sampling

- If able to directly sample $p(\mathbf{x})$,

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- Unnormalized weights $\tilde{w}_i = \frac{\tilde{p}(\mathbf{x}_i)}{q(\mathbf{x}_i)}$, samples $\mathbf{x}_i \sim q(\mathbf{x})$

$$\mathrm{E}[\mathbf{f}(\mathbf{x})] = \sum_{i=1}^{N} \frac{\tilde{w}_i}{\sum_{j=1}^{N} \tilde{w}_j} \mathbf{f}(\mathbf{x}_i). \tag{21}$$

# Approximating a Probability Density Function with Importance Sampling

- The importance sampling approximation to $p(\mathbf{x})$ given the unnormalized distribution $\tilde{p}(\mathbf{x})$, and a proposal distribution $q(\mathbf{x})$, is thus given by

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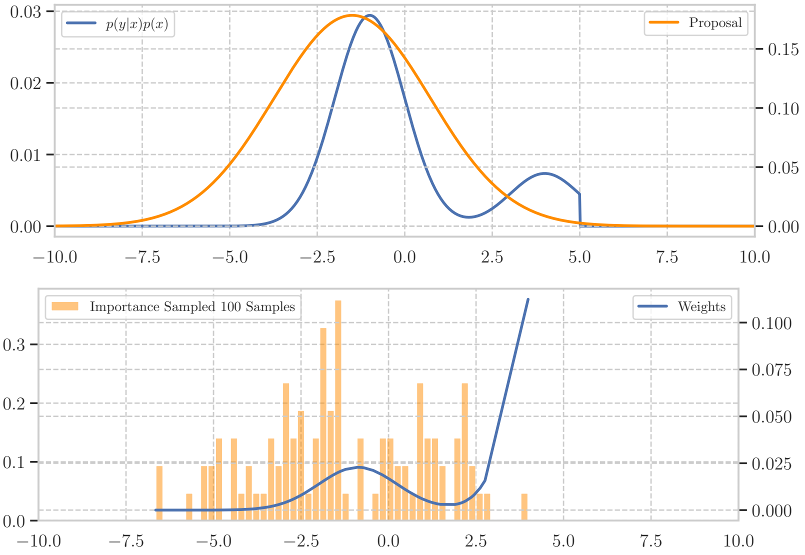
$$\boxed{w_i = \frac{\tilde{p}(\mathbf{x}_i)/q(\mathbf{x}_i)}{\sum_{j=1}^{N} \tilde{p}(\mathbf{x}_j)/q(\mathbf{x}_j)}.} \tag{23}$$

# Importance Sampling Illustration



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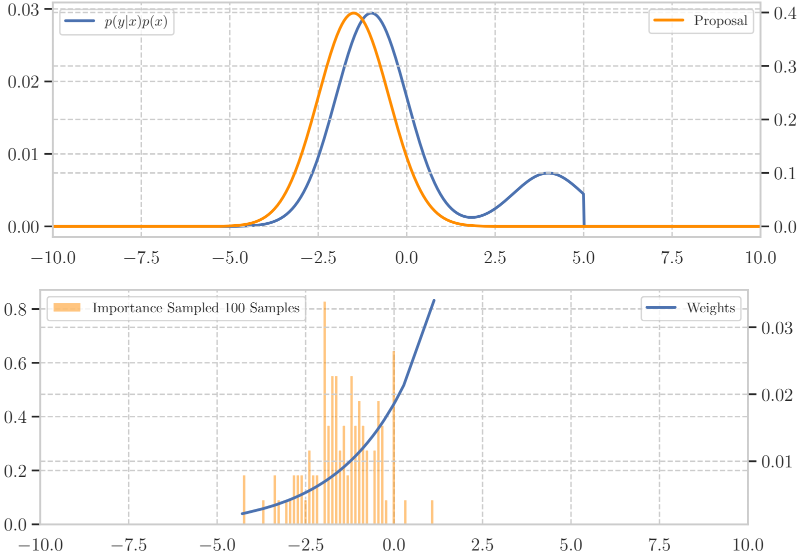
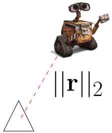
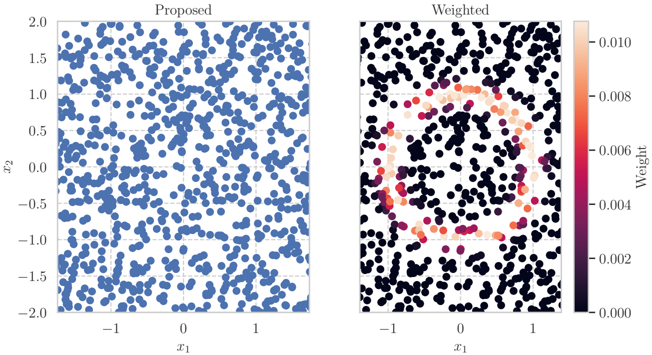


# Importance Sampling Illustration

- Single measurement.
- Uniform proposal distribution.



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- Animation!

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*Can use any Monte Carlo sampling method!*

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- Filtering distribution given by

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- By using the $N$ integrands in (37) in an arbitrary Monte Carlo solver, we develop a set of particles that approximate the sum of the integrals in (37).

# Sequential Monte Carlo with Importance Sampling

- Using importance sampling on (37) with proposal distribution $q(\mathbf{x}_k|\mathbf{x}_{i,k-1})$ gives

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- Running an MCMC method on each integral of (38), after resampling, corresponds to the *resample-move* algorithm. Only the last sample in the chain is kept.

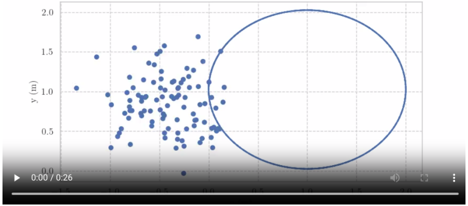
# Demo - A Simple Example

- Vector state $\mathbf{x}_k \in \mathbb{R}^2$.
- Single integrator process model,

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{u}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k). \tag{41}$$

- Range measurement to anchor in the center of the scene,

$$\mathbf{y}_k = \|\mathbf{x}_k\|_2^2 + w_k, \quad w_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}). \tag{42}$$



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draw a new set $\mathbf{x}_j$ from the discrete distribution

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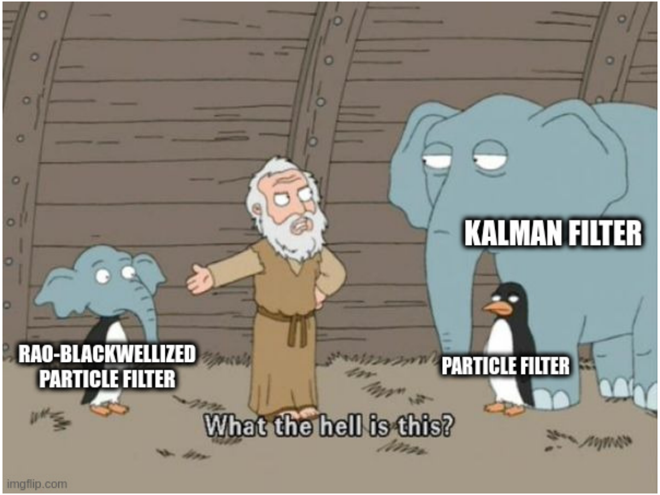
- Can cause *sample impoverishment* where particles lose diversity.
- Adaptive resampling - resample only when needed. For exaple, use effective number of particles as a threshold,

$$n_{\text{eff}} \approx \frac{1}{\sum_{i=1}^{N} {w_k^{(i)}}^2}. \tag{45}$$

# Rao-Blackwellization



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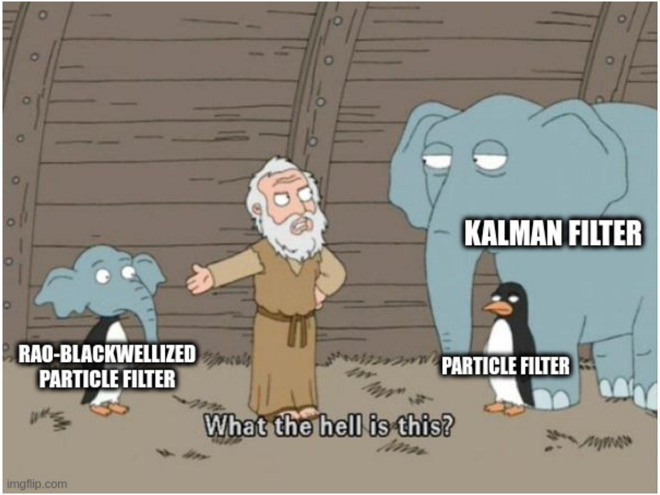


Partition state into non-Gaussian part $\mathbf{u}$ and *conditionally Gaussian* part $\mathbf{x}$.

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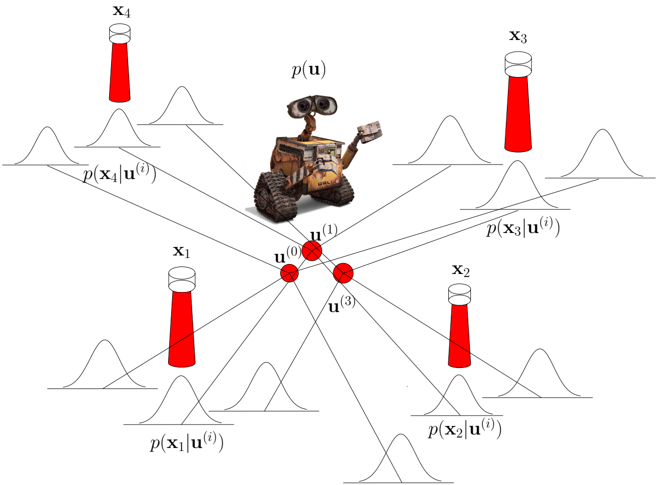


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# Rao-Blackwellization

- Models of form

$$p(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{u}_{k-1}) = \mathcal{N}(\mathbf{x}_k; \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1})), \mathbf{Q}_{k-1}(\mathbf{u}_{k-1})), \tag{46}$$

$$p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) = \mathcal{N}(\mathbf{y}_k; \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k), \mathbf{R}_k(\mathbf{u}_k)), \tag{47}$$

$$p(\mathbf{u}_k | \mathbf{u}_{k-1}) \sim \text{Any distribution.} \tag{48}$$

where $\mathbf{u}_k$ is *non-Gaussian part of the state.*

# Rao-Blackwellization

- Models of form

$$
\begin{aligned}
p(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{u}_{k-1}) &= \mathcal{N}(\mathbf{x}_k;\mathbf{f}(\mathbf{x}_{k-1},\mathbf{u}_{k-1})),\mathbf{Q}_{k-1}(\mathbf{u}_{k-1})), && (46)\\
p(\mathbf{y}_k|\mathbf{x}_k,\mathbf{u}_k) &= \mathcal{N}(\mathbf{y}_k;\mathbf{g}(\mathbf{x}_k,\mathbf{u}_k),\mathbf{R}_k(\mathbf{u}_k)), && (47)\\
p(\mathbf{u}_k|\mathbf{u}_{k-1}) &\sim \text{Any distribution.} && (48)
\end{aligned}
$$

  where $\mathbf{u}_k$ is *non-Gaussian part of the state.*

- State belief

$$
p(\mathbf{x}_k,\mathbf{u}_k|\mathbf{y}_{1:k}) = \sum_{i=1}^{N} w_k^{(i)} \delta\left(\mathbf{u}_k - \mathbf{u}_k^{(i)}\right) \mathcal{N}\left(\mathbf{x}_k;\hat{\mathbf{x}}_k^{(i)},\hat{\mathbf{P}}_k^{(i)}\right). \tag{49}
$$

# Rao-Blackwellization

- Models of form

$$p(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{u}_{k-1}) = \mathcal{N}(\mathbf{x}_k;\mathbf{f}(\mathbf{x}_{k-1},\mathbf{u}_{k-1})),\mathbf{Q}_{k-1}(\mathbf{u}_{k-1})), \tag{46}$$

$$p(\mathbf{y}_k|\mathbf{x}_k,\mathbf{u}_k) = \mathcal{N}(\mathbf{y}_k;\mathbf{g}(\mathbf{x}_k,\mathbf{u}_k),\mathbf{R}_k(\mathbf{u}_k)), \tag{47}$$

$$p(\mathbf{u}_k|\mathbf{u}_{k-1}) \sim \text{Any distribution.} \tag{48}$$

  where $\mathbf{u}_k$ is *non-Gaussian part of the state.*

- State belief

$$p(\mathbf{x}_k,\mathbf{u}_k|\mathbf{y}_{1:k}) = \sum_{i=1}^{N} w_k^{(i)}\delta\left(\mathbf{u}_k-\mathbf{u}_k^{(i)}\right)\mathcal{N}\left(\mathbf{x}_k;\hat{\mathbf{x}}_k^{(i)},\hat{\mathbf{P}}_k^{(i)}\right). \tag{49}$$

- State part $\mathbf{x}$ is conditionally Gaussian given $\mathbf{u}$,

$$p(\mathbf{x}_k|\mathbf{u}_k,\mathbf{y}_{1:k}) = \sum_{j=1}^{N} I(\mathbf{u}_j=\mathbf{u}_k)\mathcal{N}(\mathbf{x}_k|\mathbf{x}_k^{(j)},\mathbf{P}_k^{(j)}). \tag{50}$$

  where $I(*)$ is the indicator function that is equal to one if the input condition is fulfilled, and zero if not.

# Rao-Blackwellization

- The Bayes filter takes the form,

$$
\begin{aligned}
p(\mathbf{x}_k, \mathbf{u}_k | \mathbf{y}_{k-1:k}) &= \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{u}_k | \mathbf{y}_{k-1:k}) && (51) \\
&= \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{y}_k | \mathbf{u}_k, \mathbf{y}_{k-1}) p(\mathbf{u}_k | \mathbf{y}_{k-1}) && (52)
\end{aligned}
$$

# Rao-Blackwellization

- The Bayes filter takes the form,

$$p(\mathbf{x}_k, \mathbf{u}_k | \mathbf{y}_{k-1:k}) = \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{u}_k | \mathbf{y}_{k-1:k}) \tag{51}$$

$$= \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{y}_k | \mathbf{u}_k, \mathbf{y}_{k-1}) p(\mathbf{u}_k | \mathbf{y}_{k-1}) \tag{52}$$

- The predicted distribution of $\mathbf{u}$ is given by the Chapman-Kolmogorov equation,

$$p(\mathbf{u}_k | \mathbf{y}_{k-1}) = \int p(\mathbf{u}_k | \mathbf{u}_{k-1}) p(\mathbf{u}_{k-1} | \mathbf{y}_{k-1}) \mathrm{d}\mathbf{u}_{k-1} \tag{53}$$

$$= \int p(\mathbf{u}_k | \mathbf{u}_{k-1}) \sum_{i=1}^{N} w_{k-1}^{i} \delta(\mathbf{u}_{k-1} - \mathbf{u}_{k-1}^{(i)}) \mathrm{d}\mathbf{u}_{k-1} \tag{54}$$

$$= \sum_{i=1}^{N} w_{k-1}^{i} p(\mathbf{u}_k | \mathbf{u}_{k-1}^{(i)}). \tag{55}$$

# Rao-Blackwellization - Prediction step for non-Gaussian state

- Importance sampling. Define $q(\mathbf{u}_k|\mathbf{u}^i_{k-1})$ and sample a $\mathbf{u}^{(i)}_k$ with corresponding predicted weight

$$\check{w}^{(i)}_k = w^{(i-1)}_k \frac{p(\mathbf{u}^{(i)}_k|\mathbf{u}^{(i)}_{k-1})}{q(\mathbf{u}^{(i-1)}_k|\mathbf{u}^{(i)}_{k-1})}. \tag{56}$$

# Rao-Blackwellization - Prediction step for non-Gaussian state

- Importance sampling. Define $q(\mathbf{u}_k|\mathbf{u}_{k-1}^i)$ and sample a $\mathbf{u}_k^{(i)}$ with corresponding predicted weight

$$\check{w}_k^{(i)} = w_k^{(i-1)} \frac{p(\mathbf{u}_k^{(i)}|\mathbf{u}_{k-1}^{(i)})}{q(\mathbf{u}_k^{(i-1)}|\mathbf{u}_{k-1}^{(i)})}. \tag{56}$$

- The predicted distribution on $\mathbf{u}_k$ is thus

$$p(\mathbf{u}_k|\mathbf{y}_{k-1}) = \sum_{i=1}^{N} \check{w}_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}). \tag{57}$$

# Rao-Blackwellization - Correction step for non-Gaussian state

- The correction consists of updating the weights using the marginal likelihood $p(\mathbf{y}_k|\mathbf{u}_k, \mathbf{y}_{k-1})$, which is obtained by

$$p(\mathbf{y}_k|\mathbf{u}_k, \mathbf{y}_{k-1}) = \int p(\mathbf{y}_k, \mathbf{x}_k|\mathbf{u}_k, \mathbf{y}_{k-1})\mathrm{d}\mathbf{x}_k \tag{58}$$

$$= \int p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{u}_k)p(\mathbf{x}_k|\mathbf{u}_k, \mathbf{y}_{k-1})\mathrm{d}\mathbf{x}_k, \tag{59}$$

  where $p(\mathbf{x}_k|\mathbf{u}_k, \mathbf{y}_{k-1})$ is given by

$$p(\mathbf{x}_k|\mathbf{u}_k, \mathbf{y}_{k-1}) = \sum_{i=1}^{N} I(\mathbf{u}_i = \mathbf{u}_k)\mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}). \tag{60}$$

# Rao-Blackwellization - Correction step for non-Gaussian state

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$$= \int p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{u}_k) p(\mathbf{x}_k|\mathbf{u}_k, \mathbf{y}_{k-1}) \mathrm{d}\mathbf{x}_k, \tag{59}$$

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$$p(\mathbf{x}_k|\mathbf{u}_k, \mathbf{y}_{k-1}) = \sum_{i=1}^{N} I(\mathbf{u}_i = \mathbf{u}_k) \mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}). \tag{60}$$

- The likelihood (59) becomes

$$p(\mathbf{y}_k|\mathbf{u}_k, \mathbf{y}_{k-1}) = \sum_{i=1}^{N} w_{k-1}^{(i)} I(\mathbf{u}_i = \mathbf{u}_k) \int p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{u}_k) \mathcal{N}\left(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}\right) \mathrm{d}\mathbf{x}_k, \tag{61}$$

where each integral obtained as the marginal measurement mean and covariance of the Gaussian filter update.

# Rao-Blackwellization - Updated Belief on non-Gaussian State

The likelihood (61) is thus combined with (57) to give

$$p(\mathbf{u}_k|\mathbf{y}_{k-1:k}) = \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}), \tag{62}$$

with

$$\mathbf{u}_k^{(i)} \sim q(\mathbf{u}_k^{(i-1)}|\mathbf{u}_{k-1}^{(i)}) \tag{63}$$

$$w_k^{(i)} = \underbrace{w_k^{(i-1)} \frac{p(\mathbf{u}_k^{(i)}|\mathbf{u}_{k-1}^{(i)})}{q(\mathbf{u}_k^{(i-1)}|\mathbf{u}_{k-1}^{(i)})}}_{\check{w}_k^{(i)}} \int p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{u}_k) \mathcal{N}\left(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}\right) \mathrm{d}\mathbf{x}_k. \tag{64}$$

# Rao-Blackwellization - Updating Conditionally Gaussian State

- The Bayes filter (51) becomes

$$p(\mathbf{x}_k, \mathbf{u}_k | \mathbf{y}_{k-1:k}) = \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{u}_k | \mathbf{y}_{k-1:k}) \tag{65}$$

(66)

(67)

# Rao-Blackwellization - Updating Conditionally Gaussian State

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$$p(\mathbf{x}_k, \mathbf{u}_k | \mathbf{y}_{k-1:k}) = \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{u}_k | \mathbf{y}_{k-1:k}) \tag{65}$$

$$= \frac{1}{\eta} p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) \tag{66}$$

(67)

# Rao-Blackwellization - Updating Conditionally Gaussian State

- The Bayes filter (51) becomes

$$
\begin{aligned}
p(\mathbf{x}_k, \mathbf{u}_k | \mathbf{y}_{k-1:k}) &= \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{u}_k | \mathbf{y}_{k-1:k}) && (65) \\
&= \frac{1}{\eta} p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) && (66) \\
&= \frac{1}{\eta} p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) \sum_{i=1}^{N} I(\mathbf{u}_i = \mathbf{u}_k) \mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}). && (67)
\end{aligned}
$$

# Rao-Blackwellization - Updating Conditionally Gaussian State

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$$p(\mathbf{x}_k, \mathbf{u}_k|\mathbf{y}_{k-1:k}) = \frac{1}{\eta}p(\mathbf{x}_k|\mathbf{u}_k, \mathbf{y}_{k-1:k})p(\mathbf{u}_k|\mathbf{y}_{k-1:k}) \tag{65}$$

$$= \frac{1}{\eta}p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{u}_k)p(\mathbf{x}_k|\mathbf{u}_k, \mathbf{y}_{k-1}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) \tag{66}$$

$$= \frac{1}{\eta}p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{u}_k) \sum_{i=1}^{N} I(\mathbf{u}_i = \mathbf{u}_k)\mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}). \tag{67}$$

- Which simplifies to

$$p(\mathbf{x}_k, \mathbf{u}_k|\mathbf{y}_{k-1:k}) = \frac{1}{\eta} \sum_{i=1}^{N} p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{u}_k)\mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}) w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) \tag{68}$$

$$\tag{69}$$

# Rao-Blackwellization - Updating Conditionally Gaussian State

- The Bayes filter (51) becomes

$$
\begin{aligned}
p(\mathbf{x}_k, \mathbf{u}_k | \mathbf{y}_{k-1:k}) &= \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{u}_k | \mathbf{y}_{k-1:k}) && (65) \\
&= \frac{1}{\eta} p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) && (66) \\
&= \frac{1}{\eta} p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) \sum_{i=1}^{N} I(\mathbf{u}_i = \mathbf{u}_k) \mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}). && (67)
\end{aligned}
$$

- Which simplifies to

$$
\begin{aligned}
p(\mathbf{x}_k, \mathbf{u}_k | \mathbf{y}_{k-1:k}) &= \frac{1}{\eta} \sum_{i=1}^{N} p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) \mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}) w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) && (68) \\
&= \frac{1}{\eta} \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) \underbrace{p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) \mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)})}_{\text{Predict/Correct for each particle's } \mathbf{x}}, && (69)
\end{aligned}
$$

# Rao-Blackwellization - Updating Conditionally Gaussian State

- The Bayes filter (51) becomes

$$p(\mathbf{x}_k, \mathbf{u}_k | \mathbf{y}_{k-1:k}) = \frac{1}{\eta} p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1:k}) p(\mathbf{u}_k | \mathbf{y}_{k-1:k}) \tag{65}$$

$$= \frac{1}{\eta} p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) p(\mathbf{x}_k | \mathbf{u}_k, \mathbf{y}_{k-1}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) \tag{66}$$

$$= \frac{1}{\eta} p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) \sum_{i=1}^{N} I(\mathbf{u}_i = \mathbf{u}_k) \mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)}) \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}). \tag{67}$$

- Which simplifies to

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$$= \frac{1}{\eta} \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{u}_k - \mathbf{u}_k^{(i)}) \underbrace{p(\mathbf{y}_k | \mathbf{x}_k, \mathbf{u}_k) \mathcal{N}(\mathbf{x}_k; \check{\mathbf{x}}_k^{(i)}, \check{\mathbf{P}}_k^{(i)})}_{\text{Predict/Correct for each particle's } \mathbf{x}}, \tag{69}$$

where the non-Gaussian state update for $w_k^{(i)}, \mathbf{u}_k^{(i)}$ is given by (64).

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