Calibration Doc

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October 22, 2025

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1 State Transition Matrices

1.1 IMU Kinematics

1.1.1 Linearized Error Dynamics

The state transition matrix is computed by first considering the linearized error dynamics. For the IMU kinematics, a left perturbation is used on the extended pose, and an additive error is used on the bias and landmark state. The state is thus given by $\mathcal{X} = \left(\mathbf{C}_{ab}, \mathbf{v}_a^{ba}, \mathbf{r}_a^{ba}, \mathbf{b}^{\text{acc}}, \mathbf{b}^{\omega}\right) \in SE_2(3) \times \mathbb{R}^6$. The linearized dynamics matrices and the state transition matrices for the calibration parameters and landmarks are zero and identity, respectively, as these states are assumed constant. For the IMU, the left perturbation on $SE_2(3)$ is given by

$$\mathbf{T} = \delta\mathbf{T}\bar{\mathbf{T}} \tag{1}$$

$$= \begin{bmatrix} \delta\mathbf{C} & \delta\mathbf{v} & \delta\mathbf{r} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{C}} & \bar{\mathbf{v}} & \bar{\mathbf{r}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} \mathbf{C} & \mathbf{v} & \mathbf{r} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \delta\mathbf{C}\bar{\mathbf{C}} & \delta\mathbf{C}\bar{\mathbf{v}} + \delta\mathbf{v} & \delta\mathbf{C}\bar{\mathbf{r}} + \delta\mathbf{r} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

such that isolating for each perturbation yields

$$\delta\mathbf{C} = \mathbf{C}\bar{\mathbf{C}}^{\mathsf{T}}, \tag{4}$$

$$\delta\mathbf{v} = \mathbf{v} - \delta\mathbf{C}\bar{\mathbf{v}}, \tag{5}$$

$$\delta\mathbf{r} = \mathbf{r} - \delta\mathbf{C}\bar{\mathbf{r}}, \tag{6}$$

which implies

$$
\begin{aligned}
\mathbf{C} &= \delta\mathbf{C}\bar{\mathbf{C}} && (7)\\
\mathbf{v} &= \delta\mathbf{v} + \delta\mathbf{C}\bar{\mathbf{v}} && (8)\\
\mathbf{r} &= \delta\mathbf{r} + \delta\mathbf{C}\bar{\mathbf{r}} && (9)\\
& && (10)
\end{aligned}
$$

The bias errors are given by

$$
\begin{aligned}
\delta\mathbf{b}^{\text{acc}} &= \mathbf{b}^{\text{acc}} - \bar{\mathbf{b}}^{\text{acc}}, && (11)\\
\delta\mathbf{b}^{\omega} &= \mathbf{b}^{\omega} - \bar{\mathbf{b}}^{\omega}, && (12)
\end{aligned}
$$

which implies

$$
\begin{aligned}
\mathbf{b}^{\text{acc}} &= \bar{\mathbf{b}}^{\text{acc}} + \delta\mathbf{b}^{\text{acc}} && (13)\\
\mathbf{b}^{\omega} &= \bar{\mathbf{b}}^{\omega} + \delta\mathbf{b}^{\omega}. && (14)
\end{aligned}
$$

The calibration parameter errors are given by

$$
\begin{aligned}
\delta\mathbf{C}_{bc} &= \mathbf{C}_{bc}\bar{\mathbf{C}}_{bc}^{\mathsf{T}} && (15)\\
\mathbf{r}_b^{bc} &= \mathbf{r}_b^{bc} - \bar{\mathbf{r}}_b^{bc}. && (16)
\end{aligned}
$$

The IMU kinematics are given by

$$
\begin{aligned}
\dot{\mathbf{C}} &= \mathbf{C}\boldsymbol{\omega}^{\times} = \mathbf{C}(\mathbf{u}^{\omega} - \mathbf{b}^{\omega} - \mathbf{w}^{\omega})^{\times} && (17)\\
\dot{\mathbf{v}} &= \mathbf{C}(\mathbf{u}^{\text{acc}} - \mathbf{b}^{\text{acc}} - \mathbf{w}^{\text{acc}}) + \mathbf{g} && (18)\\
\dot{\mathbf{r}} &= \mathbf{v} && (19)\\
\dot{\mathbf{b}}^{\omega} &= \mathbf{w}^{\omega} && (20)\\
\dot{\mathbf{b}}^{\text{acc}} &= \mathbf{w}^{\text{acc}}. && (21)
\end{aligned}
$$

The linearized error dynamics are derived by taking the error derivatives, then linearizing. For the IMU DCM this becomes

$$
\begin{aligned}
\delta\dot{\mathbf{C}} &= \dot{\mathbf{C}}\bar{\mathbf{C}}^{\mathsf{T}} + \mathbf{C}\dot{\bar{\mathbf{C}}}^{\mathsf{T}} && (22)\\
&= \mathbf{C}(\mathbf{u}^{\omega} - \mathbf{b}^{\omega} - \mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}} + \mathbf{C}\left(\bar{\mathbf{C}}(\mathbf{u}^{\omega} - \bar{\mathbf{b}}^{\omega})\right)^{\mathsf{T}} && (23)\\
&= \mathbf{C}(\mathbf{u}^{\omega} - \mathbf{b}^{\omega} - \mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}} - \mathbf{C}(\mathbf{u}^{\omega} - \bar{\mathbf{b}}^{\omega})\bar{\mathbf{C}}^{\mathsf{T}} && (24)\\
&= \delta\mathbf{C}\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}}. && (25)
\end{aligned}
$$

Linearizing in terms of $\delta\boldsymbol{\xi}^{\phi}$ yields

$$\begin{aligned}
\delta\dot{\boldsymbol{\xi}}^{\phi\times} &= \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}\right)\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}} && (26)\\
&= \bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}} && (27)\\
&= \left(\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})\right)^{\times}, && (28)\\
\delta\dot{\boldsymbol{\xi}}^{\phi} &= \bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega}) && (29)\\
&= -\bar{\mathbf{C}}\delta\mathbf{b}^{\omega} - \bar{\mathbf{C}}\delta\mathbf{w}^{\omega} && (30)\\
&= \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{C}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\xi}^{\phi}\\ \delta\boldsymbol{\xi}^{\mathrm{v}}\\ \delta\boldsymbol{\xi}^{\mathrm{r}}\\ \delta\boldsymbol{\xi}^{b_{\omega}}\\ \delta\boldsymbol{\xi}^{b_{\mathrm{acc}}}\end{bmatrix} + \begin{bmatrix}-\bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\mathbf{w}^{\omega}\\ \delta\mathbf{w}^{\mathrm{acc}}\\ \delta\mathbf{w}^{b_{\omega}}\\ \delta\mathbf{w}^{b_{\mathrm{acc}}}\end{bmatrix}. && (31)
\end{aligned}$$

The velocity error dynamics are given by

$$\begin{aligned}
\delta\dot{\mathbf{v}} &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} - \delta\mathbf{C}\bar{\mathbf{v}}) && (32)\\
&= \mathbf{C}(\mathbf{u}^{\mathrm{acc}} - \mathbf{b}^{\mathrm{acc}} - \mathbf{w}^{\mathrm{acc}}) + \mathbf{g} - \delta\mathbf{C}\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{v}} - \delta\mathbf{C}\bar{\mathbf{C}}^{\mathsf{T}}(\mathbf{u}^{\mathrm{acc}} - \bar{\mathbf{b}}^{\mathrm{acc}}) && (33)\\
&= \delta\mathbf{C}\bar{\mathbf{C}}(\mathbf{u}^{\mathrm{acc}} - \mathbf{b}^{\mathrm{acc}} - \mathbf{w}^{\mathrm{acc}}) + \mathbf{g} - \delta\mathbf{C}\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{v}} - \delta\mathbf{C}\bar{\mathbf{C}}^{\mathsf{T}}(\mathbf{u}^{\mathrm{acc}} - \bar{\mathbf{b}}^{\mathrm{acc}}) && (34)
\end{aligned}$$

where IMU kinematics are substituted directly into the first term and the last two terms are obtained using the product rule. Isolating the bias in terms of the nominal one and the error yields

$$\begin{aligned}
\delta\dot{\mathbf{v}} &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} - \delta\mathbf{C}\bar{\mathbf{v}}) && (35)\\
&= \mathbf{C}(\mathbf{u}^{\mathrm{acc}} - \mathbf{b}^{\mathrm{acc}} - \mathbf{w}^{\mathrm{acc}}) + \mathbf{g} - \delta\mathbf{C}\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{v}} - \delta\mathbf{C}(\bar{\mathbf{C}}^{\mathsf{T}}(\mathbf{u}^{\mathrm{acc}} - \bar{\mathbf{b}}^{\mathrm{acc}}) + \mathbf{g}) && (36)\\
&= \delta\mathbf{C}\bar{\mathbf{C}}(\mathbf{u}^{\mathrm{acc}} - \mathbf{b}^{\mathrm{acc}} - \mathbf{w}^{\mathrm{acc}}) + \mathbf{g} - \delta\mathbf{C}\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{v}} - \delta\mathbf{C}(\bar{\mathbf{C}}^{\mathsf{T}}(\mathbf{u}^{\mathrm{acc}} - \bar{\mathbf{b}}^{\mathrm{acc}}) + \mathbf{g}) && (37)\\
&= \delta\mathbf{C}\bar{\mathbf{C}}(-\delta\mathbf{b}^{\mathrm{acc}} - \delta\mathbf{w}^{\mathrm{acc}}) + (\mathbf{1} - \delta\mathbf{C})\mathbf{g} - \delta\mathbf{C}\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{v}}. && (38)
\end{aligned}$$

Linearizing and neglecting small terms yields

$$\begin{aligned}
\delta\boldsymbol{\xi}^{\mathrm{v}} &= \bar{\mathbf{C}}(-\delta\mathbf{b}^{\mathrm{acc}} - \delta\mathbf{w}^{\mathrm{acc}}) - \delta\boldsymbol{\xi}^{\phi\times}\mathbf{g} - \bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{v}} && (39)\\
&= \bar{\mathbf{C}}(-\delta\mathbf{b}^{\mathrm{acc}} - \delta\mathbf{w}^{\mathrm{acc}}) - \delta\boldsymbol{\xi}^{\phi\times}\mathbf{g} + \bar{\mathbf{C}}(\delta\mathbf{b}^{\omega} + \delta\mathbf{w}^{\omega})^{\times}\bar{\mathbf{C}}^{\mathsf{T}}\bar{\mathbf{v}} && (40)\\
&= \bar{\mathbf{C}}(-\delta\mathbf{b}^{\mathrm{acc}} - \delta\mathbf{w}^{\mathrm{acc}}) - \delta\boldsymbol{\xi}^{\phi\times}\mathbf{g} + \left(\bar{\mathbf{C}}(\delta\mathbf{b}^{\omega} + \delta\mathbf{w}^{\omega})\right)^{\times}\bar{\mathbf{v}} && (41)\\
&= -\bar{\mathbf{C}}\delta\mathbf{b}^{\mathrm{acc}} - \bar{\mathbf{C}}\delta\mathbf{w}^{\mathrm{acc}} - \mathbf{g}^{\times}\delta\boldsymbol{\xi}^{\phi} - \bar{\mathbf{v}}^{\times}\left(\bar{\mathbf{C}}(\delta\mathbf{b}^{\omega} + \delta\mathbf{w}^{\omega})\right) && (42)\\
&= -\bar{\mathbf{C}}\delta\mathbf{b}^{\mathrm{acc}} - \bar{\mathbf{C}}\delta\mathbf{w}^{\mathrm{acc}} - \mathbf{g}^{\times}\delta\boldsymbol{\xi}^{\phi} - \bar{\mathbf{v}}^{\times}\bar{\mathbf{C}}\delta\mathbf{b}^{\omega} - \bar{\mathbf{v}}^{\times}\bar{\mathbf{C}}\delta\mathbf{w}^{\omega} && (43)\\
&= \begin{bmatrix}\mathbf{g}^{\times} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{v}}^{\times}\bar{\mathbf{C}} & -\bar{\mathbf{C}}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\xi}^{\phi}\\ \delta\boldsymbol{\xi}^{\mathrm{v}}\\ \delta\boldsymbol{\xi}^{\mathrm{r}}\\ \delta\boldsymbol{\xi}^{b_{\omega}}\\ \delta\boldsymbol{\xi}^{b_{\mathrm{acc}}}\end{bmatrix} + \begin{bmatrix}-\bar{\mathbf{v}}^{\times}\bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{C}}\end{bmatrix}\begin{bmatrix}\delta\mathbf{w}^{\omega}\\ \delta\mathbf{w}^{\mathrm{acc}}\\ \delta\mathbf{w}^{b_{\omega}}\\ \delta\mathbf{w}^{b_{\mathrm{acc}}}\end{bmatrix}. && (44)
\end{aligned}$$

The position error derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{r} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} - \delta\mathbf{C}\bar{\mathbf{r}}) \tag{45}$$

$$= \mathbf{v} - \delta\dot{\mathbf{C}}\bar{\mathbf{r}} - \delta\mathbf{C}\bar{\mathbf{v}} \tag{46}$$

$$= \delta\mathbf{v} + \delta\mathbf{C}\bar{\mathbf{v}} - \left(\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})\right)^{\times}\bar{\mathbf{r}} - \delta\mathbf{C}\bar{\mathbf{v}} \tag{47}$$

$$= \delta\mathbf{v} - \left(\bar{\mathbf{C}}(-\delta\mathbf{b}^{\omega} - \delta\mathbf{w}^{\omega})\right)^{\times}\bar{\mathbf{r}} \tag{48}$$

$$= \delta\mathbf{v} + \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}}(\delta\mathbf{b}^{\omega} + \delta\mathbf{w}^{\omega}) \tag{49}$$

Linearizing yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{\xi}^{\mathrm{r}} = \delta\mathbf{v} + \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}}(\delta\mathbf{b}^{\omega} + \delta\mathbf{w}^{\omega}) \tag{50}$$

$$= \delta\boldsymbol{\xi}^{\mathrm{v}} + \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}}\delta\mathbf{b}^{\omega} + \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}}\delta\mathbf{w}^{\omega} \tag{51}$$

$$= \begin{bmatrix}\mathbf{0} & \mathbf{1} & \mathbf{0} & \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\xi}^{\phi}\\ \delta\boldsymbol{\xi}^{\mathrm{v}}\\ \delta\boldsymbol{\xi}^{\mathrm{r}}\\ \delta\boldsymbol{\xi}^{b_{\omega}}\\ \delta\boldsymbol{\xi}^{b_{\mathrm{acc}}}\end{bmatrix} + \begin{bmatrix}\bar{\mathbf{r}}^{\times}\bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\mathbf{w}^{\omega}\\ \delta\mathbf{w}^{\mathrm{acc}}\\ \delta\mathbf{w}^{b_{\omega}}\\ \delta\mathbf{w}^{b_{\mathrm{acc}}}\end{bmatrix}. \tag{52}$$

For the accelerometer bias,

$$\delta\mathbf{b}^{\mathrm{acc}} = \mathbf{b}^{\mathrm{acc}} - \bar{\mathbf{b}}^{\mathrm{acc}}, \tag{53}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{b}^{\mathrm{acc}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{b}^{\mathrm{acc}} - \bar{\mathbf{b}}^{\mathrm{acc}}\right) \tag{54}$$

$$= \mathbf{w}^{b_{\mathrm{acc}}}. \tag{55}$$

Linearizing,

$$\delta\boldsymbol{\xi}^{b_{\mathrm{acc}}} = \delta\mathbf{w}^{b_{\mathrm{acc}}} \tag{56}$$

$$= \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\xi}^{\phi}\\ \delta\boldsymbol{\xi}^{\mathrm{v}}\\ \delta\boldsymbol{\xi}^{\mathrm{r}}\\ \delta\boldsymbol{\xi}^{b_{\omega}}\\ \delta\boldsymbol{\xi}^{b_{\mathrm{acc}}}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\delta\mathbf{w}^{\omega}\\ \delta\mathbf{w}^{\mathrm{acc}}\\ \delta\mathbf{w}^{b_{\omega}}\\ \delta\mathbf{w}^{b_{\mathrm{acc}}}\end{bmatrix}. \tag{57}$$

The same steps hold for the gyroscope bias. The full linearized error dynamics are thus given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{\xi} = \mathbf{F}_{\mathrm{c}}\delta\boldsymbol{\xi} + \mathbf{L}_{\mathrm{c}}\delta\mathbf{w} \tag{58}$$

$$= \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{C}} & \mathbf{0}\\ \mathbf{g}^{\times} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{v}}^{\times}\bar{\mathbf{C}} & -\bar{\mathbf{C}}\\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\xi}^{\phi}\\ \delta\boldsymbol{\xi}^{\mathrm{v}}\\ \delta\boldsymbol{\xi}^{\mathrm{r}}\\ \delta\boldsymbol{\xi}^{b_{\omega}}\\ \delta\boldsymbol{\xi}^{b_{\mathrm{acc}}}\end{bmatrix} + \begin{bmatrix}-\bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ -\bar{\mathbf{v}}^{\times}\bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{C}}\\ \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\delta\mathbf{w}^{\omega}\\ \delta\mathbf{w}^{\mathrm{acc}}\\ \delta\mathbf{w}^{b_{\omega}}\\ \delta\mathbf{w}^{b_{\mathrm{acc}}}\end{bmatrix}. \tag{59}$$

1.1.2 State Transition Matrix from ODE

The state transition matrix arises in the discretization of a linear time-varying (LTV) system in continuous time. Specifically, given a system without inputs,

$$\dot{\mathbf{x}} = \mathbf{F}_{\mathrm{c}}(t)\mathbf{x}(t) \tag{60}$$

the equivalent discrete-time model is given by

$$\mathbf{x}(t) = \mathbf{\Phi}_(t)\mathbf{x}_0 \tag{61}$$

The state transition matrix $\mathbf{\Phi}_(t)$ is of interest. By requiring $\mathbf{\Phi}_(t)$ to satisfy the following conditions,

$$\mathbf{\Phi}(0) = \mathbf{1} \tag{62}$$

$$\dot{\mathbf{\Phi}}(t) = \mathbf{F}_{\mathrm{c}}(t)\mathbf{\Phi}(t), \tag{63}$$

the resultant expression for $\mathbf{x}(t)$ in (61) satisfies the ODE (60). This yields an equivalent discrete-time model, without any approximations. For the IMU kinematics, this takes the following form,

$$\mathbf{\Phi}(0) = \mathbf{1}, \tag{64}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \mathbf{\Phi}_{1,1} & \mathbf{\Phi}_{1,2} & \mathbf{\Phi}_{1,3} & \mathbf{\Phi}_{1,4} & \mathbf{\Phi}_{1,5} \\ \mathbf{\Phi}_{2,1} & \mathbf{\Phi}_{2,2} & \mathbf{\Phi}_{2,3} & \mathbf{\Phi}_{2,4} & \mathbf{\Phi}_{2,5} \\ \mathbf{\Phi}_{3,1} & \mathbf{\Phi}_{3,2} & \mathbf{\Phi}_{3,3} & \mathbf{\Phi}_{3,4} & \mathbf{\Phi}_{3,5} \\ \mathbf{\Phi}_{4,1} & \mathbf{\Phi}_{4,2} & \mathbf{\Phi}_{4,3} & \mathbf{\Phi}_{4,4} & \mathbf{\Phi}_{4,5} \\ \mathbf{\Phi}_{5,1} & \mathbf{\Phi}_{5,2} & \mathbf{\Phi}_{5,3} & \mathbf{\Phi}_{5,4} & \mathbf{\Phi}_{5,5} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{g}^{\times} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{v}}^{\times}\bar{\mathbf{C}} & -\bar{\mathbf{C}} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{1,1} & \mathbf{\Phi}_{1,2} & \mathbf{\Phi}_{1,3} & \mathbf{\Phi}_{1,4} & \mathbf{\Phi}_{1,5} \\ \mathbf{\Phi}_{2,1} & \mathbf{\Phi}_{2,2} & \mathbf{\Phi}_{2,3} & \mathbf{\Phi}_{2,4} & \mathbf{\Phi}_{2,5} \\ \mathbf{\Phi}_{3,1} & \mathbf{\Phi}_{3,2} & \mathbf{\Phi}_{3,3} & \mathbf{\Phi}_{3,4} & \mathbf{\Phi}_{3,5} \\ \mathbf{\Phi}_{4,1} & \mathbf{\Phi}_{4,2} & \mathbf{\Phi}_{4,3} & \mathbf{\Phi}_{4,4} & \mathbf{\Phi}_{4,5} \\ \mathbf{\Phi}_{5,1} & \mathbf{\Phi}_{5,2} & \mathbf{\Phi}_{5,3} & \mathbf{\Phi}_{5,4} & \mathbf{\Phi}_{5,5} \end{bmatrix} \tag{65}$$

It immediately follows for the last two rows that

$$\begin{bmatrix} \mathbf{\Phi}_{4,1} & \mathbf{\Phi}_{4,2} & \mathbf{\Phi}_{4,3} & \mathbf{\Phi}_{4,4} & \mathbf{\Phi}_{4,5} \\ \mathbf{\Phi}_{5,1} & \mathbf{\Phi}_{5,2} & \mathbf{\Phi}_{5,3} & \mathbf{\Phi}_{5,4} & \mathbf{\Phi}_{5,5} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \tag{66}$$

such that

$$\mathbf{\Phi}(0) = \mathbf{1}, \tag{67}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \mathbf{\Phi}_{1,1} & \mathbf{\Phi}_{1,2} & \mathbf{\Phi}_{1,3} & \mathbf{\Phi}_{1,4} & \mathbf{\Phi}_{1,5} \\ \mathbf{\Phi}_{2,1} & \mathbf{\Phi}_{2,2} & \mathbf{\Phi}_{2,3} & \mathbf{\Phi}_{2,4} & \mathbf{\Phi}_{2,5} \\ \mathbf{\Phi}_{3,1} & \mathbf{\Phi}_{3,2} & \mathbf{\Phi}_{3,3} & \mathbf{\Phi}_{3,4} & \mathbf{\Phi}_{3,5} \\ \mathbf{\Phi}_{4,1} & \mathbf{\Phi}_{4,2} & \mathbf{\Phi}_{4,3} & \mathbf{\Phi}_{4,4} & \mathbf{\Phi}_{4,5} \\ \mathbf{\Phi}_{5,1} & \mathbf{\Phi}_{5,2} & \mathbf{\Phi}_{5,3} & \mathbf{\Phi}_{5,4} & \mathbf{\Phi}_{5,5} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{g}^{\times} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{v}}^{\times}\bar{\mathbf{C}} & -\bar{\mathbf{C}} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{1,1} & \mathbf{\Phi}_{1,2} & \mathbf{\Phi}_{1,3} & \mathbf{\Phi}_{1,4} & \mathbf{\Phi}_{1,5} \\ \mathbf{\Phi}_{2,1} & \mathbf{\Phi}_{2,2} & \mathbf{\Phi}_{2,3} & \mathbf{\Phi}_{2,4} & \mathbf{\Phi}_{2,5} \\ \mathbf{\Phi}_{3,1} & \mathbf{\Phi}_{3,2} & \mathbf{\Phi}_{3,3} & \mathbf{\Phi}_{3,4} & \mathbf{\Phi}_{3,5} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{68}$$

Multiplying out the block matrices in (68) and substituting the zeros/identities for the last two rows yields

$$\dot{\mathbf{\Phi}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{g}^{\times}\mathbf{\Phi}_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{g}^{\times}\mathbf{\Phi}_{1,4} - \bar{\mathbf{v}}^{\times}\bar{\mathbf{C}} & \mathbf{g}^{\times}\mathbf{\Phi}_{1,5} - \bar{\mathbf{C}} \\ \mathbf{\Phi}_{2,1} & \mathbf{\Phi}_{2,2} & \mathbf{\Phi}_{2,3} & \mathbf{\Phi}_{2,4} + \bar{\mathbf{r}}^{\times}\bar{\mathbf{C}} & \mathbf{\Phi}_{2,5} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{69}$$

Considering zero derivative blocks and initial conditions yields additional structure for $\mathbf{\Phi}$,

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{1,4} & \mathbf{0} \\ \mathbf{\Phi}_{2,1} & \mathbf{1} & \mathbf{0} & \mathbf{\Phi}_{2,4} & \mathbf{\Phi}_{2,5} \\ \mathbf{\Phi}_{3,1} & \mathbf{\Phi}_{3,2} & \mathbf{\Phi}_{3,3} & \mathbf{\Phi}_{3,4} & \mathbf{\Phi}_{3,5} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{70}$$

The following subblocks may then be solved for as

- Given that $\dot{\mathbf{\Phi}}_{3,3} = \mathbf{\Phi}_{2,3} = \mathbf{0}$, $\mathbf{\Phi}_{3,3} = \mathbf{1}$.
- Given that $\dot{\mathbf{\Phi}}_{3,2} = \mathbf{\Phi}_{2,2} = \mathbf{1}$, $\mathbf{\Phi}_{3,2}(0) = \mathbf{0}$, therefore $\mathbf{\Phi}_{3,2} = t\mathbf{1}$.
- Given that $\dot{\mathbf{\Phi}}_{2,1} = \mathbf{g}^\times \mathbf{\Phi}_{1,1} = \mathbf{g}^\times \mathbf{1}$, and that $\mathbf{\Phi}_{2,1}(0) = \mathbf{0}$, then $\mathbf{\Phi}_{2,1} = t\mathbf{g}^\times$.
- Given that $\dot{\mathbf{\Phi}}_{3,1} = \mathbf{\Phi}_{2,1} = t\mathbf{g}^\times$, and that $\mathbf{\Phi}_{3,1}(0) = \mathbf{0}$, then $\mathbf{\Phi}_{3,1} = \frac{1}{2}t^2\mathbf{g}^\times$.
- Given that $\dot{\mathbf{\Phi}}_{1,4} = -\bar{\mathbf{C}}$, $\mathbf{\Phi}_{1,4}(0) = 0$, then $\mathbf{\Phi}_{1,4} = -\int_0^t \bar{\mathbf{C}}(t)\mathrm{d}t$.
- Given that $\dot{\mathbf{\Phi}}_{2,5} = \mathbf{g}^\times \mathbf{\Phi}_{1,5} - \bar{\mathbf{C}}$, and $\mathbf{\Phi}_{1,5} = \mathbf{0}$, and that $\mathbf{\Phi}_{2,5}(0) = \mathbf{0}$, then $\mathbf{\Phi}_{2,5} = -\int_0^t \bar{\mathbf{C}}(t)\mathrm{d}t$.

Therefore the structure of $\mathbf{\Phi}$ is updated as

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & -\int_0^t \bar{\mathbf{C}}(t)\mathrm{d}t & \mathbf{0} \\ t\mathbf{g}^\times & \mathbf{1} & \mathbf{0} & \mathbf{\Phi}_{2,4} & -\int_0^t \bar{\mathbf{C}}(t)\mathrm{d}t \\ \frac{1}{2}t^2\mathbf{g}^\times & t\mathbf{1} & \mathbf{1} & \mathbf{\Phi}_{3,4} & \mathbf{\Phi}_{3,5} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{71}$$

The remaining subblocks evolve according to the following ODEs,

$$\dot{\mathbf{\Phi}}_{2,4} = \mathbf{g}^\times \mathbf{\Phi}_{1,4} - \mathbf{v}^\times \bar{\mathbf{C}} = -\mathbf{g}^\times \int_0^t \bar{\mathbf{C}}(t)\mathrm{d}t - \bar{\mathbf{v}}^\times \bar{\mathbf{C}}, \tag{72}$$

$$\dot{\mathbf{\Phi}}_{3,4} = \mathbf{\Phi}_{2,4} + \bar{\mathbf{r}}^\times \bar{\mathbf{C}}, \tag{73}$$

$$\dot{\mathbf{\Phi}}_{3,5} = \mathbf{\Phi}_{2,5} = -\int_0^t \bar{\mathbf{C}}(t)\mathrm{d}t. \tag{74}$$

Therefore, the $\mathbf{\Phi}$ subblocks are given by the following. Special attention must be paid to the integration bounds and what each nested integrand is a function of.

$$\mathbf{\Phi}_{2,4} = -\mathbf{g}^\times \int_0^t \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau - \int_0^t \bar{\mathbf{v}}^\times(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}\tau, \tag{75}$$

$$\mathbf{\Phi}_{3,4} = -\mathbf{g}^\times \int_0^t \int_0^\theta \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau\mathrm{d}\theta - \int_0^t \int_0^\theta \bar{\mathbf{v}}^\times(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}\tau\mathrm{d}\theta + \int_0^t \bar{\mathbf{r}}^\times(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}t \tag{76}$$

$$\mathbf{\Phi}_{3,5} = -\int_0^t \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau. \tag{77}$$

1.2 SE(3) Kinematics

The continuous $SE(3)$ kinematics are given by

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab}(\boldsymbol{\omega}_b^{ba} + \mathbf{w}_1)^\times \tag{78}$$
$$\dot{\mathbf{r}}_a^{ba} = \mathbf{C}_{ab}(\mathbf{v}_b^{ba} + \mathbf{w}_2). \tag{79}$$

The rotation part is linearized as

$$\begin{aligned}
\delta\dot{\mathbf{C}}_b &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{C}_b\bar{\mathbf{C}}_b^\mathsf{T}) && (80)\\
&= \dot{\mathbf{C}}_b\bar{\mathbf{C}}_b^\mathsf{T} + \mathbf{C}_b\dot{\bar{\mathbf{C}}}_b^\mathsf{T} && (81)\\
&= \dot{\mathbf{C}}_b\bar{\mathbf{C}}_b^\mathsf{T} + \mathbf{C}_b\dot{\bar{\mathbf{C}}}_b^\mathsf{T} && (82)\\
&= \mathbf{C}_b(\boldsymbol{\omega}_b^{ba} + \mathbf{w}_1)^\times\bar{\mathbf{C}}_b^\mathsf{T} + \mathbf{C}_b\left(\bar{\mathbf{C}}_b\boldsymbol{\omega}_b^{ba^\times}\right)^\mathsf{T} && (83)\\
&= \mathbf{C}_b\mathbf{w}_1^\times\bar{\mathbf{C}}_b^\mathsf{T} && (84)\\
&= \delta\mathbf{C}_b\bar{\mathbf{C}}_b\mathbf{w}_1^\times\bar{\mathbf{C}}_b^\mathsf{T} && (85)\\
&\approx (\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}_{b})\bar{\mathbf{C}}_b\mathbf{w}_1^\times\bar{\mathbf{C}}_b^\mathsf{T} && (86)\\
&\approx \bar{\mathbf{C}}_b\mathbf{w}_1^\times\bar{\mathbf{C}}_b^\mathsf{T} && (87)\\
&= (\bar{\mathbf{C}}_b\mathbf{w}_1)^\times && (88)\\
\delta\dot{\boldsymbol{\xi}}^\phi &= \bar{\mathbf{C}}_b\mathbf{w}_1. && (89)
\end{aligned}$$

The position part is linearized as

$$\begin{aligned}
\delta\dot{\mathbf{r}}_b &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_b - \delta\mathbf{C}_b\bar{\mathbf{r}}_b\right) && (90)\\
&= \dot{\mathbf{r}}_b - \delta\dot{\mathbf{C}}_b\bar{\mathbf{r}}_b - \delta\mathbf{C}_b\dot{\bar{\mathbf{r}}}_b && (91)\\
&= \mathbf{C}_b(\mathbf{v}_b + \mathbf{w}_2) - (\bar{\mathbf{C}}_b\mathbf{w}_1)^\times\bar{\mathbf{r}}_b - \delta\mathbf{C}_b\bar{\mathbf{C}}_b\mathbf{v}_b && (92)\\
&= \delta\mathbf{C}_b\bar{\mathbf{C}}_b(\mathbf{v}_b + \mathbf{w}_2) - (\bar{\mathbf{C}}_b\mathbf{w}_1)^\times\bar{\mathbf{r}}_b - \delta\mathbf{C}_b\bar{\mathbf{C}}_b\mathbf{v}_b && (93)\\
&= \delta\mathbf{C}_b\bar{\mathbf{C}}_b\mathbf{w}_2 - (\bar{\mathbf{C}}_b\mathbf{w}_1)^\times\bar{\mathbf{r}}_b && (94)\\
&\approx \left(\mathbf{1} + \delta\boldsymbol{\xi}_b^{\phi\times}\right)\bar{\mathbf{C}}_b\mathbf{w}_2 - (\bar{\mathbf{C}}_b\mathbf{w}_1)^\times\bar{\mathbf{r}}_b && (95)\\
&\approx \bar{\mathbf{C}}_b\mathbf{w}_2 - (\bar{\mathbf{C}}_b\mathbf{w}_1)^\times\bar{\mathbf{r}}_b && (96)\\
&= \bar{\mathbf{C}}_b\mathbf{w}_2 + \bar{\mathbf{r}}_b^\times\bar{\mathbf{C}}_b\mathbf{w}_1 && (97)\\
& && (98)
\end{aligned}$$

The continuous-time linearized error dynamics are therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\delta\boldsymbol{\xi}^\phi\\ \delta\boldsymbol{\xi}^\mathrm{r}\end{bmatrix} = \begin{bmatrix}\mathbf{0}^{3\times3} & \mathbf{0}^{3\times3}\\ \mathbf{0}^{3\times3} & \mathbf{0}^{3\times3}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\xi}^\phi\\ \delta\boldsymbol{\xi}^\mathrm{r}\end{bmatrix} + \begin{bmatrix}\bar{\mathbf{C}}_b & \mathbf{0}^{3\times3}\\ \bar{\mathbf{r}}^\times\bar{\mathbf{C}}_b & \bar{\mathbf{C}}\end{bmatrix}\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{bmatrix}. \tag{99}$$

The state transition matrix is therefore identity. We can also start with

$$\dot{\mathbf{T}} = \mathbf{T}\boldsymbol{\varpi}^\wedge. \tag{100}$$

Using a left perturbation yields

$$\mathbf{T} = \delta\mathbf{T}\bar{\mathbf{T}} \tag{101}$$
$$\delta\mathbf{T} = \mathbf{T}\bar{\mathbf{T}}^{-1}, \tag{102}$$

where taking the time derivative yields

$$\delta\dot{\mathbf{T}} = \dot{\mathbf{T}}\bar{\mathbf{T}}^{-1} + \mathbf{T}\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{T}}^{-1}. \tag{103}$$

The quantity $\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{T}}^{-1}$ is obtained by considering

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{T}\mathbf{T}^{-1}) = \dot{\mathbf{T}}\mathbf{T}^{-1} + \mathbf{T}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}^{-1} = \mathbf{0} \tag{104}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}^{-1} = -\mathbf{T}^{-1}\dot{\mathbf{T}}\mathbf{T}^{-1}. \tag{105}$$

Therefore,

$$\delta\dot{\mathbf{T}} = \dot{\mathbf{T}}\bar{\mathbf{T}}^{-1} - \mathbf{T}\bar{\mathbf{T}}^{-1}\dot{\bar{\mathbf{T}}}\bar{\mathbf{T}}^{-1} \tag{106}$$
$$= \mathbf{T}\boldsymbol{\varpi}^{\wedge}\bar{\mathbf{T}}^{-1} - \mathbf{T}\bar{\mathbf{T}}^{-1}\bar{\mathbf{T}}\boldsymbol{\varpi}^{\wedge}\bar{\mathbf{T}}^{-1} \tag{107}$$
$$= \mathbf{T}\boldsymbol{\varpi}^{\wedge}\bar{\mathbf{T}}^{-1} - \mathbf{T}\boldsymbol{\varpi}^{\wedge}\bar{\mathbf{T}}^{-1} = \mathbf{0}. \tag{108}$$

Since the extrinsics calibration parameters are assumed constant, their state transition matrices are also identity.

2 Measurement Models

2.1 Camera Measurement Model

The projection measurement model, given a landmark resolved in the world frame $\boldsymbol{\ell}_a$, the relative landmark position in the robot body frame is given by

$$\mathbf{r}_b^{b\ell} = \mathbf{C}_{ab}^{\mathsf{T}}\left(\boldsymbol{\ell}_a - \mathbf{r}_a\right), \tag{1}$$

and the camera pinhole projection model is obtained by projecting this quantity to the image plane,

$$\mathbf{r}_b^{b\ell} = \mathbf{C}_{ab}^{\mathsf{T}}\left(\boldsymbol{\ell}_a - \mathbf{r}_a\right), \tag{2}$$
$$\mathbf{y}_c = \mathbf{s}(\mathbf{r}_b^{b\ell}) = \frac{1}{r_{b,3}^{b\ell}}\begin{bmatrix} r_{b,1}^{b\ell} \\ r_{b,2}^{b\ell} \end{bmatrix}. \tag{3}$$

To derive the measurement model Jacobian, the subscripts are dropped for convenience such that

$$\mathbf{p}(\mathbf{C}, \mathbf{r}, \boldsymbol{\ell}) = \mathbf{C}^{\mathsf{T}}\left(\boldsymbol{\ell} - \mathbf{r}\right) \tag{4}$$
$$\mathbf{y}_c = \mathbf{s}(\mathbf{p}) = \frac{1}{p_3}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}. \tag{5}$$

The chain rule is used to derive the Jacobians, with

$$\frac{\partial}{\partial \mathbf{p}}\mathbf{s}(\mathbf{p}) = \begin{bmatrix} \frac{1}{p_3} & 0 & -p_1/p_3^2 \\ 0 & \frac{1}{p_3} & -p_2/p_3^2 \end{bmatrix}, \tag{6}$$

and the Lie group Jacobians derived using a left perturbation. An additive error is used for the landmark. The error definitions and resultant state expressions are

$$\begin{aligned} \delta\mathbf{C} &= \mathbf{C}\bar{\mathbf{C}}^\mathsf{T}, && (7)\\ \delta\mathbf{r} &= \mathbf{r} - \delta\mathbf{C}\bar{\mathbf{r}}, && (8)\\ \mathbf{C} &= \delta\mathbf{C}\bar{\mathbf{C}} && (9)\\ \mathbf{r} &= \delta\mathbf{r} + \delta\mathbf{C}\bar{\mathbf{r}} && (10)\\ \delta\boldsymbol{\ell} &= \boldsymbol{\ell} - \bar{\boldsymbol{\ell}} && (11)\\ \boldsymbol{\ell} &= \bar{\boldsymbol{\ell}} + \delta\boldsymbol{\ell}. && (12) \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{p}(\mathbf{C},\mathbf{r},\boldsymbol{\ell}) &= \mathbf{C}^\mathsf{T}\left(\boldsymbol{\ell} - \mathbf{r}\right) && (13)\\ &= (\delta\mathbf{C}\bar{\mathbf{C}})^\mathsf{T}\left((\bar{\boldsymbol{\ell}} + \delta\boldsymbol{\ell}) - (\delta\mathbf{r} + \delta\mathbf{C}\bar{\mathbf{r}})\right) && (14)\\ &= \bar{\mathbf{C}}^\mathsf{T}\delta\mathbf{C}^\mathsf{T}\left((\bar{\boldsymbol{\ell}} + \delta\boldsymbol{\ell}) - \delta\mathbf{r} + \delta\mathbf{C}\bar{\mathbf{r}}\right) && (15)\\ &= \bar{\mathbf{C}}^\mathsf{T}(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times})^\mathsf{T}\left((\bar{\boldsymbol{\ell}} + \delta\boldsymbol{\ell}) - \left(\delta\boldsymbol{\xi}^{\mathrm{r}} + \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}\right)\bar{\mathbf{r}}\right)\right) && (16)\\ &= \bar{\mathbf{C}}^\mathsf{T}\left((\bar{\boldsymbol{\ell}} + \delta\boldsymbol{\ell}) - \left(\delta\boldsymbol{\xi}^{\mathrm{r}} + \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}\right)\bar{\mathbf{r}}\right)\right) - \bar{\mathbf{C}}^\mathsf{T}\delta\boldsymbol{\xi}^{\phi\times}\left((\bar{\boldsymbol{\ell}} + \delta\boldsymbol{\ell}) - \left(\delta\boldsymbol{\xi}^{\mathrm{r}} + \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\phi\times}\right)\bar{\mathbf{r}}\right)\right) && (17)\\ &= \bar{\mathbf{C}}^\mathsf{T}\left(\bar{\boldsymbol{\ell}} - \bar{\mathbf{r}}\right) + \bar{\mathbf{C}}^\mathsf{T}\left(\delta\boldsymbol{\ell} - \delta\boldsymbol{\xi}^{\mathrm{r}} - \delta\boldsymbol{\xi}^{\phi\times}\bar{\mathbf{r}}\right) - \bar{\mathbf{C}}^\mathsf{T}\delta\boldsymbol{\xi}^{\phi\times}\left(\bar{\boldsymbol{\ell}} - \bar{\mathbf{r}}\right) && (18)\\ &= \bar{\mathbf{p}} - \bar{\mathbf{C}}^\mathsf{T}\delta\boldsymbol{\xi}^{\mathrm{r}} + \bar{\mathbf{C}}^\mathsf{T}\delta\boldsymbol{\ell} - \bar{\mathbf{C}}^\mathsf{T}\delta\boldsymbol{\xi}^{\phi\times}\bar{\boldsymbol{\ell}} && (19)\\ &= \bar{\mathbf{p}} - \bar{\mathbf{C}}^\mathsf{T}\delta\boldsymbol{\xi}^{\mathrm{r}} + \bar{\mathbf{C}}^\mathsf{T}\delta\boldsymbol{\ell} + \bar{\mathbf{C}}^\mathsf{T}\bar{\boldsymbol{\ell}}^{\times}\delta\boldsymbol{\xi}^{\phi} && (20)\\ &= \bar{\mathbf{p}} + \begin{bmatrix} \bar{\mathbf{C}}^\mathsf{T}\bar{\boldsymbol{\ell}}^{\times} & -\bar{\mathbf{C}}^\mathsf{T} & \bar{\mathbf{C}}^\mathsf{T} \end{bmatrix}\begin{bmatrix} \delta\boldsymbol{\xi}^{\phi} \\ \delta\boldsymbol{\xi}^{\mathrm{r}} \\ \delta\boldsymbol{\ell} \end{bmatrix}, && (21)\\ \frac{D\mathbf{p}}{D\mathcal{X}} &= \begin{bmatrix} \bar{\mathbf{C}}^\mathsf{T}\bar{\boldsymbol{\ell}}^{\times} & -\bar{\mathbf{C}}^\mathsf{T} & \bar{\mathbf{C}}^\mathsf{T} \end{bmatrix}, && (22) \end{aligned}$$

and the left perturbation measurement Jacobian is given by

$$\frac{D}{D\mathcal{X}}\mathbf{y}_c = \frac{\partial \mathbf{s}}{\partial \mathbf{p}}\frac{D\mathbf{p}}{D\mathcal{X}}. \tag{23}$$

2.2 Global Pose - SE(3)

The sensors involved are a local sensor with reference frame $\mathcal{F}_b$, a global sensor with reference frame $\mathcal{F}_c$, and the global world frame $\mathcal{F}_a$. The state vector is given by $\mathcal{X} = \left(\mathbf{C}_{ab}, \mathbf{r}_a^{ba}, \mathbf{C}_{bc}, \mathbf{r}_b^{cb}, \tau_{bc}.\right)$. The timestamps are such that $t_b = t_c + \tau_{bc}$. The local sensor provides relative pose transformations

$\mathbf{T}_{b_k b_{k-1}}$. The global sensor provides global pose information, $\mathbf{C}_{ac}, \mathbf{r}_a^{ca}$. A left perturbation is used. The states live in $SE(3)$ The perturbation on $SE(3)$ is given by

$$\mathbf{T} = \delta\mathbf{T}\bar{\mathbf{T}} \tag{24}$$
$$= \begin{bmatrix} \delta\mathbf{C} & \delta\mathbf{r} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{C}} & \bar{\mathbf{r}} \\ 0 & 1 \end{bmatrix} \tag{25}$$
$$\begin{bmatrix} \mathbf{C} & \mathbf{r} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \delta\mathbf{C}\bar{\mathbf{C}} & \delta\mathbf{C}\bar{\mathbf{r}} + \delta\mathbf{r} \\ 0 & 1 \end{bmatrix}, \tag{26}$$

The error definitions and resultant state expressions are

$$\delta\mathbf{C} = \mathbf{C}\bar{\mathbf{C}}^{\mathsf{T}}, \tag{27}$$
$$\delta\mathbf{r} = \mathbf{r} - \delta\mathbf{C}\bar{\mathbf{r}}, \tag{28}$$
$$\mathbf{C} = \delta\mathbf{C}\bar{\mathbf{C}}, \tag{29}$$
$$\mathbf{r} = \delta\mathbf{r} + \delta\mathbf{C}\bar{\mathbf{r}}. \tag{30}$$

To alleviate notational burden, the following two state vectors are used interchangeably. The first notation is used when kinematics are important, the second when linearizing and using error Jacobians. $\mathcal{X} = \left(\mathbf{C}_{ab}, \mathbf{r}_a^{ba}, \mathbf{C}_{bc}, \mathbf{r}_b^{cb}, \tau_{bc}.\right)$, and $\mathcal{X} = (\mathbf{C}_b, \mathbf{r}_b, \mathbf{C}_c, \mathbf{r}_c, \tau)$. Thus error definitions and resultant state expressions for this problem are

$$\delta\mathbf{C}_b = \mathbf{C}_b\bar{\mathbf{C}}_b^{\mathsf{T}}, \tag{31}$$
$$\delta\mathbf{r}_b = \mathbf{r}_b - \delta\mathbf{C}_b\bar{\mathbf{r}}_b, \tag{32}$$
$$\mathbf{C}_b = \delta\mathbf{C}_b\bar{\mathbf{C}}_b, \tag{33}$$
$$\mathbf{r}_b = \delta\mathbf{r}_b + \delta\mathbf{C}_b\bar{\mathbf{r}}_b \tag{34}$$
$$\delta\mathbf{C}_c = \mathbf{C}_c\bar{\mathbf{C}}_c^{\mathsf{T}}, \tag{35}$$
$$\delta\mathbf{r}_c = \mathbf{r}_c - \delta\mathbf{C}_c\bar{\mathbf{r}}_c, \tag{36}$$
$$\mathbf{C}_c = \delta\mathbf{C}_c\bar{\mathbf{C}}_c, \tag{37}$$
$$\mathbf{r}_c = \delta\mathbf{r}_c + \delta\mathbf{C}_c\bar{\mathbf{r}}_c. \tag{38}$$

2.2.1 No Time Offset Case

For no time offset, the global sensor measures

$$g_{\text{syn}}(\mathcal{X}) = g_{\text{syn}}(\mathbf{T}_{ab_k}, \mathbf{T}_{bc}) \tag{39}$$
$$= \begin{pmatrix} \mathbf{C}_{ac} \\ \mathbf{r}_a^{ca} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{ab}\mathbf{C}_{bc} \\ \mathbf{r}_a^{ba} + \mathbf{r}_a^{cb} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{ab}\mathbf{C}_{bc} \\ \mathbf{r}_a^{ba} + \mathbf{C}_{ab}\mathbf{r}_b^{cb} \end{pmatrix}. \tag{40}$$

With shortened notation, the same expression reads

$$\mathbf{y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{ac} \\ \mathbf{r}_a^{ca} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_b\mathbf{C}_c \\ \mathbf{r}_b + \mathbf{C}_b\mathbf{r}_c \end{pmatrix}. \tag{41}$$

A Taylor series expansion of $\mathbf{Y}_1$ yields

$$
\begin{aligned}
\mathbf{Y}_1 &= \mathbf{C}_b\mathbf{C}_c && (42)\\
&= (\mathbf{1} + \delta\boldsymbol{\xi}_b^{\phi\times})\bar{\mathbf{C}}_b(\mathbf{1} + \delta\boldsymbol{\xi}_c^{\phi\times})\bar{\mathbf{C}}_c && (43)\\
&\approx \bar{\mathbf{Y}}_1 + \delta\boldsymbol{\xi}_b^{\phi\times}\bar{\mathbf{C}}_b\bar{\mathbf{C}}_c + \bar{\mathbf{C}}_b\delta\boldsymbol{\xi}_c^{\phi\times}\bar{\mathbf{C}}_c, && (44)\\
&= \bar{\mathbf{Y}}_1 + (\delta\boldsymbol{\xi}_b^{\phi} + \mathrm{Ad}_{\bar{\mathbf{C}}_b}\delta\boldsymbol{\xi}_c^{\phi})^{\times}\bar{\mathbf{C}}_b\bar{\mathbf{C}}_c && (45)\\
&= \bar{\mathbf{Y}}_1 + (\delta\boldsymbol{\xi}_b^{\phi} + \bar{\mathbf{C}}_b\delta\boldsymbol{\xi}_c^{\phi})^{\times}\bar{\mathbf{C}}_b\bar{\mathbf{C}}_c && (46)\\
(\mathbf{1} + \delta\boldsymbol{\xi}_y^{\times})\bar{\mathbf{Y}}_1 &= \bar{\mathbf{Y}}_1 + (\delta\boldsymbol{\xi}_b^{\phi} + \bar{\mathbf{C}}_b\delta\boldsymbol{\xi}_c^{\phi})^{\times}\bar{\mathbf{C}}_b\bar{\mathbf{C}}_c && (47)\\
\delta\boldsymbol{\xi}_y &= \delta\boldsymbol{\xi}_b^{\phi} + \bar{\mathbf{C}}_b\delta\boldsymbol{\xi}_c^{\phi} && (48)\\
&= \begin{bmatrix}\mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_b & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\xi}_b^{\phi}\\ \delta\boldsymbol{\xi}_b^{\mathrm{r}}\\ \delta\boldsymbol{\xi}_c^{\phi}\\ \delta\boldsymbol{\xi}_c^{\mathrm{r}}\end{bmatrix}. && (49)
\end{aligned}
$$

A Taylor expansion of $\mathbf{y}_2$ yields

$$
\begin{aligned}
\mathbf{y}_2 &= \mathbf{r}_b + \mathbf{C}_b\mathbf{r}_c && (50)\\
&= \delta\mathbf{r}_b + \delta\mathbf{C}_b\bar{\mathbf{r}}_b + \delta\mathbf{C}_b\bar{\mathbf{C}}_b(\delta\mathbf{r}_c + \delta\mathbf{C}_c\bar{\mathbf{r}}_c) && (51)\\
&\approx \delta\mathbf{r}_b + \left(\mathbf{1} + \delta\boldsymbol{\xi}_b^{\phi\times}\right)\bar{\mathbf{r}}_b + \left(\mathbf{1} + \delta\boldsymbol{\xi}_b^{\phi\times}\right)\bar{\mathbf{C}}_b\left(\delta\mathbf{r}_c + \left(\mathbf{1} + \delta\boldsymbol{\xi}_c^{\phi\times}\right)\bar{\mathbf{r}}_c\right) && (52)\\
&\approx \delta\mathbf{r}_b + \bar{\mathbf{r}}_b + \delta\boldsymbol{\xi}_b^{\phi\times}\bar{\mathbf{r}}_b + \bar{\mathbf{C}}_b\left(\delta\mathbf{r}_c + \left(\mathbf{1} + \delta\boldsymbol{\xi}_c^{\phi\times}\right)\bar{\mathbf{r}}_c\right) + \delta\boldsymbol{\xi}_b^{\phi\times}\bar{\mathbf{C}}_b\bar{\mathbf{r}}_c && (53)\\
&= \delta\mathbf{r}_b + \bar{\mathbf{r}}_b + \delta\boldsymbol{\xi}_b^{\phi\times}\bar{\mathbf{r}}_b + \bar{\mathbf{C}}_b\delta\mathbf{r}_c + \bar{\mathbf{C}}_b\left(\mathbf{1} + \delta\boldsymbol{\xi}_c^{\phi\times}\right)\bar{\mathbf{r}}_c + \delta\boldsymbol{\xi}_b^{\phi\times}\bar{\mathbf{C}}_b\bar{\mathbf{r}}_c && (54)\\
&= \bar{\mathbf{y}}_2 + \delta\mathbf{r}_b + \delta\boldsymbol{\xi}_b^{\phi\times}\bar{\mathbf{r}}_b + \bar{\mathbf{C}}_b\delta\mathbf{r}_c + \bar{\mathbf{C}}_b\delta\boldsymbol{\xi}_c^{\phi\times}\bar{\mathbf{r}}_c + \delta\boldsymbol{\xi}_b^{\phi\times}\bar{\mathbf{C}}_b\bar{\mathbf{r}}_c && (55)\\
&= \bar{\mathbf{y}}_2 + \delta\mathbf{r}_b - \bar{\mathbf{r}}_b^{\times}\delta\boldsymbol{\xi}_b^{\phi} + \bar{\mathbf{C}}_b\delta\mathbf{r}_c - \bar{\mathbf{C}}_b\bar{\mathbf{r}}_c^{\times}\delta\boldsymbol{\xi}_c^{\phi} - (\bar{\mathbf{C}}_b\bar{\mathbf{r}}_c)^{\times}\delta\boldsymbol{\xi}_b^{\phi} && (56)\\
&= \bar{\mathbf{y}}_2 + \begin{bmatrix}-\left(\bar{\mathbf{r}}_b^{\times} + (\bar{\mathbf{C}}_b\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_b\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_b\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\xi}_b^{\phi}\\ \delta\boldsymbol{\xi}_b^{\mathrm{r}}\\ \delta\boldsymbol{\xi}_c^{\phi}\\ \delta\boldsymbol{\xi}_c^{\mathrm{r}}\end{bmatrix}. && (57)
\end{aligned}
$$

The overall measurement model Jacobian is thus given by

$$
\frac{Dg_{\text{syn}}}{D\mathcal{X}} = \begin{bmatrix}\mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_b & \mathbf{0}\\ -\left(\bar{\mathbf{r}}_b^{\times} + (\bar{\mathbf{C}}_b\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_b\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_b\end{bmatrix}. \tag{58}
$$

2.2.2 Time Offset Case

This is derived following the style of [1]. Due to the delay in timestamping the measurement, the recorded camera timestamp is larger than the actual camera timestamp, with time delay

$$
t_c^k = t_b^k + \tau_{cb}, \tag{59}
$$

Therefore, when the reported timestamp is given by t_c^k, this corresponds to

$$t_b^k = t_c^k - \tau_{cb}, \tag{60}$$

in the base clock. Example: Camera image is captured, delay is 100ms. Once it reaches the IMU, it corresponds to 100ms in the past. The camera time t_c^k is given. This can also be written

$$t_b^k = t_c^k + \tau_{bc}. \tag{61}$$

We assume that global sensor fired at same time as local sensor, with a time delay in the measurement being timestamped. The Jacobians w.r.t. spatial extrinsics remain the same. In considering the timestamps, the measurement model is written as The measurement model is

$$\begin{aligned}
\mathbf{y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{ac} \\ \mathbf{r}_a^{ca} \end{pmatrix}\Bigg|_{t_b^k} &= \begin{pmatrix} \mathbf{C}_b^{\left(t_c^k - \tau_{cb}\right)} \mathbf{C}_c \\ \mathbf{r}_b^{\left(t_c^k - \tau_{cb}\right)} + \mathbf{C}_b^{\left(t_c^k - \tau_{cb}\right)} \mathbf{r}_c \end{pmatrix} && (62) \\
&= \begin{pmatrix} \mathbf{C}_b^{\left(t_c^k - \tau_{cb}\right)} \mathbf{C}_c \\ \mathbf{r}_b^{\left(t_c^k - \tau_{cb}\right)} + \mathbf{C}_b^{\left(t_c^k - \tau_{cb}\right)} \mathbf{r}_c \end{pmatrix} && (63) \\
&= \begin{pmatrix} \mathbf{C}_b^{t_c^k} \operatorname{Exp}(-\boldsymbol{\omega}_k \tau_{cb}) \mathbf{C}_c \\ \mathbf{r}_b^{t_c^k} - \mathbf{v}_k \tau_{cb} + \mathbf{C}_b^{t_c^k} \operatorname{Exp}(-\boldsymbol{\omega}_k \tau_{cb}) \mathbf{r}_c \end{pmatrix} && (64)
\end{aligned}$$

Taking a Taylor series expansion of $\mathbf{Y}_1$,

$$\begin{aligned}
\mathbf{Y}_1 &= \bar{\mathbf{C}}_b^{\left(t_c^k - \tau_{cb} - \delta\tau\right)} \bar{\mathbf{C}}_c && (65) \\
&= \bar{\mathbf{C}}_b^{\left(t_c^k - \tau_{cb}\right)} \operatorname{Exp}(-\boldsymbol{\omega}_k \delta\tau) \bar{\mathbf{C}}_c && (66) \\
&= \bar{\mathbf{C}}_b^{t_b^k} \operatorname{Exp}(-\boldsymbol{\omega}_k \delta\tau) \bar{\mathbf{C}}_c && (67) \\
&= \bar{\mathbf{C}}_{b_k} \operatorname{Exp}(-\boldsymbol{\omega}_k \delta\tau) \bar{\mathbf{C}}_c && (68) \\
&= \operatorname{Exp}(-\mathrm{Ad}_{\bar{\mathbf{C}}_{b_k}} \boldsymbol{\omega}_k \delta\tau) \bar{\mathbf{C}}_{b_k} \mathbf{C}_c && (69) \\
&= \operatorname{Exp}(-\bar{\mathbf{C}}_{b_k} \boldsymbol{\omega}_k \delta\tau) \bar{\mathbf{C}}_{b_k} \bar{\mathbf{C}}_c && (70) \\
\frac{D\mathbf{Y}_1}{D\tau_{cb}} &= -\bar{\mathbf{C}}_{b_k} \boldsymbol{\omega}_k. && (71) \\
& && (72)
\end{aligned}$$

As for the second measurement,

$$\begin{aligned}
\mathbf{y}_2 &= \mathbf{r}_b^{t_c^k} - \mathbf{v}_k \tau_{cb} + \mathbf{C}_b^{t_c^k} \operatorname{Exp}(-\boldsymbol{\omega}_k \tau_{cb}) \mathbf{r}_c && (73) \\
&= \mathbf{r}_b^{t_c^k} - \mathbf{v}_k (\bar{\tau}_{cb} + \delta\tau) + \mathbf{C}_b^{t_c^k} \operatorname{Exp}(-\boldsymbol{\omega}_k (\bar{\tau}_{cb} + \delta\tau)) \mathbf{r}_c && (74) \\
&\approx \mathbf{r}_b^{t_c^k} - \mathbf{v}_k (\bar{\tau}_{cb} + \delta\tau) + \mathbf{C}_b^{t_c^k} \operatorname{Exp}(-\boldsymbol{\omega}_k \bar{\tau}_{cb}) \operatorname{Exp}(-\mathbf{J}_{\mathrm{r}, -\boldsymbol{\omega}_k \bar{\tau}_{cb}} \boldsymbol{\omega}_k \delta\tau) \mathbf{r}_c && (75) \\
&\approx \mathbf{r}_b^{t_c^k} - \mathbf{v}_k (\bar{\tau}_{cb} + \delta\tau) + \mathbf{C}_b^{t_c^k} \operatorname{Exp}(-\boldsymbol{\omega}_k \bar{\tau}_{cb}) \left(\mathbf{1} - (\mathbf{J}_{\mathrm{r}, -\boldsymbol{\omega}_k \bar{\tau}_{cb}} \boldsymbol{\omega}_k \delta\tau)^\times\right) \mathbf{r}_c && (76) \\
&= \bar{\mathbf{y}}_2 - \mathbf{v}_k \delta\tau + \mathbf{C}_b^{t_c^k} \operatorname{Exp}(-\boldsymbol{\omega}_k \bar{\tau}_{cb}) \mathbf{r}_c^\times \mathbf{J}_{\mathrm{r}, -\boldsymbol{\omega}_k \bar{\tau}_{cb}} \boldsymbol{\omega}_k \delta\tau && (77) \\
&= \bar{\mathbf{y}}_2 - \mathbf{v}_k \delta\tau + \bar{\mathbf{C}}_{b_k} \mathbf{r}_c^\times \mathbf{J}_{\mathrm{r}, -\boldsymbol{\omega}_k \bar{\tau}_{cb}} \boldsymbol{\omega}_k \delta\tau && (78)
\end{aligned}$$

For a small time offset/angular velocity, the group Jacobian becomes identity. Then,

$$\begin{aligned}\mathbf{y}_2 &= \bar{\mathbf{y}}_2 - \mathbf{v}_k\delta\tau + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\mathbf{J}_{\mathrm{r},-\boldsymbol{\omega}_k\bar{\tau}_{cb}}\boldsymbol{\omega}_k\delta\tau && (79)\\ &= \bar{\mathbf{y}}_2 + (-\mathbf{v}_k\delta\tau + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k)\delta\tau. && (80)\end{aligned}$$

3 Observability Analysis

3.1 Linearization-Based Observability Analysis

Given a discrete-time nonlinear system

$$\begin{aligned}\mathcal{X}_{k+1} &= f(\mathcal{X}_k, \mathbf{u}_k, \mathbf{w}_k), \quad \mathbf{w}_k \sim \mathbf{Q}_k, && (1)\\ \mathbf{y}_k &= \mathbf{g}(\mathcal{X}_k, \mathbf{v}_k), \quad \mathbf{v}_k \sim \mathbf{R}_k, && (2)\end{aligned}$$

a transformation $\mathcal{T}(\mathcal{X}_k, \boldsymbol{\alpha})$ is sought that yields the same measurements $\mathbf{u}_k$ and $\mathbf{y}_k$. The parameter $\boldsymbol{\alpha}$ parametrizes the transformation $\mathcal{T}(\mathcal{X}_k, \boldsymbol{\alpha})$. It can, for instance, be the scale parameter for the case of a scale unobservability.

Formally, "invariance" is required [2]. For the dynamics, requiring $\mathcal{T}(\mathcal{X}_{k+1}, \boldsymbol{\alpha}) = \mathbf{f}(\mathcal{T}(\mathcal{X}_k, \boldsymbol{\alpha}), \mathbf{u}_k, \mathbf{w}_k)$ yields the requirement

$$\mathcal{T}(f(\mathcal{X}_k, \mathbf{u}_k, \mathbf{w}_k), \boldsymbol{\alpha}) = f\left(\mathcal{T}(\mathcal{X}_k, \boldsymbol{\alpha}), \mathbf{u}_k, \mathbf{w}_k\right). \tag{3}$$

Furthermore, the measurements remain unchanged,

$$\mathbf{g}(\mathcal{X}_k, \mathbf{v}_k) = \mathbf{g}(\mathcal{T}(f(\mathcal{X}_k), \boldsymbol{\alpha}), \mathbf{v}_k). \tag{4}$$

Given the observability matrix defined by

$$\mathbf{O} = \begin{bmatrix} \mathbf{C}_k \\ \mathbf{C}_{k+1}\mathbf{A}_k \\ \mathbf{C}_{k+2}\mathbf{A}_{k+1}\mathbf{A}_k \\ \vdots \\ \mathbf{C}_{k+n_x-1}\mathbf{A}_{k+n_x-2}\ldots\mathbf{A}_k \end{bmatrix}, \tag{5}$$

where the Jacobians are given in the Lie group sense by $\mathbf{C}_k = \frac{D\mathbf{g}}{D\mathcal{X}}\big|_{\mathcal{X}_k,\mathbf{u}_k,\mathbf{0}}$ and $\mathbf{A}_k = \frac{Df}{D\mathcal{X}}\big|_{\mathcal{X}_k,\mathbf{0}}$, the columns of the Lie group Jacobian $\frac{D}{D\boldsymbol{\alpha}}\mathcal{T}(\mathcal{X},\mathbf{0})\big|_{\mathcal{X}=\mathcal{X}_0}$ lie in the nullspace of $\mathbf{O}$ [2, Prop. 1],

$$\frac{D}{D\boldsymbol{\alpha}}\mathcal{T}(\mathcal{X},\mathbf{0})\bigg|_{\mathcal{X}=\mathcal{X}_0} \in \mathcal{N}(\mathbf{O}). \tag{6}$$

Typically, a linearization-based analysis is carried out that yields $\mathcal{N}(\mathbf{O})$, from which $\mathcal{T}(\mathcal{X}, \boldsymbol{\alpha})$ must be reconstructed. Defining $\mathbf{N} \in \mathbb{R}^{n_x \times \dim \mathcal{N}(\mathbf{O})}$, where n_x is the number of d.o.f. of $\mathcal{X}$, $\mathcal{T}(\mathcal{X}, \boldsymbol{\alpha})$,

$$\begin{aligned}\mathcal{T}(\mathcal{X}, \boldsymbol{\alpha}) &\approx \mathcal{T}(\mathcal{X}, \mathbf{0}) \oplus \frac{D}{D\boldsymbol{\alpha}}\mathcal{T}(\mathcal{X},\mathbf{0})\bigg|_{\mathcal{X}=\mathcal{X}_0}\boldsymbol{\alpha} && (7)\\ &= \mathcal{T}(\mathcal{X}, \mathbf{0}) \oplus \mathbf{N}\boldsymbol{\alpha}. && (8)\end{aligned}$$

In this way, the $\mathcal{T}(\mathcal{X}, \boldsymbol{\alpha})$ may be recovered from the linearization-based observability analysis. There are some caveats,

- Technically, the reconstructed transformation is only valid for a small $\boldsymbol{\alpha}$. However, if the result satisfies the invariance conditions (3) and (4), this does not matter.
- For a given $\mathcal{T}(\mathcal{X}, \boldsymbol{\alpha})$, there's no guarantee $\mathbf{N} = \frac{D}{D\boldsymbol{\alpha}} \mathcal{T}(\mathcal{X}, \mathbf{0})\big|_{\mathcal{X}=\mathcal{X}_0}$. However, span($\mathbf{N}$) = span $\left(\frac{D}{D\boldsymbol{\alpha}} \mathcal{T}(\mathcal{X}, \mathbf{0})\big|_{\mathcal{X}=\mathcal{X}_0} \right)$, and the reconstructed transformation (8) will be the same.

The observability matrix may alternatively be written as Therefore, the observability matrix for the nonlinear Lie group case [2] can also be written as

$$
\mathbf{O} = \begin{bmatrix} \mathbf{C}_k \\ \mathbf{C}_{k+1} \boldsymbol{\Phi}(t_{k+1}, t_k) \\ \mathbf{C}_{k+2} \boldsymbol{\Phi}(t_{k+2}, t_k) \\ \vdots \\ \mathbf{C}_{k+n_x-1} \boldsymbol{\Phi}(t_{k+n_x-1}, t_k) \end{bmatrix}. \tag{9}
$$

The steps for an observability analysis are then

1. Construct an error definition, $\boldsymbol{\xi} = \mathcal{X} \ominus \mathcal{X}$.
2. Linearize to yield $\dot{\boldsymbol{\xi}} = \mathbf{A}(\mathcal{X}, \mathbf{u})\boldsymbol{\xi} + \mathbf{B}(\mathcal{X}, \mathbf{u})$.
3. Solve for the state transition matrix by solving (**??**), (**??**). Note that this *not* the same as computing $\exp(\mathbf{A}\Delta t)$.
4. Construct a single subblock of the observability matrix $\mathbf{M}_i = \mathbf{C}_i \boldsymbol{\Phi}(t_i, t_k), i > k$. The nullspace $\mathcal{O}$ of the whole matrix $\mathcal{O}$ is sought. It will correspond to the *time-independent* portion of $\mathcal{M}_i$.
5. Examine $\mathbf{M}_i$ to obtain its nullspace $\mathcal{N}(\mathbf{M}_i)$, $\mathbf{M}_i \mathbf{N} = \mathbf{0}$ with $\mathcal{N}$ spanned by the columns $\mathbf{n}_j$ of $\mathbf{N}$. This step is nontrivial, as the columns of $\mathbf{M}_i$ have to analytically be examined for combinations that yield $\mathbf{0}$. Quantities that do not vary with time are allowed to be used in the constructed nullspace vectors. *Caveat.* The quantities at the zero'th timestep, such as $\mathbf{C}_0$, $\mathbf{v}_0$, $\mathbf{r}_0$, are allowed to be used.
6. Substitute in the corresponding time-dependent quantities for the ones at the 0'th timestep. The argument is that we did this whole analysis for *any arbitrary starting point along the trajectory*.
7. Reconstruct the nonlinear transformation using (8).

3.2 VINS on Wheels: IMU + Camera

An observability analysis for the monocular camera and IMU case is presented in [3] and reproduced here. The camera measurement model is used, and the overall state is given by the IMU state and a landmark, $\mathcal{X} = (\mathcal{X}_{\text{IMU}}, \boldsymbol{\ell}) = (\mathbf{C}_{ab}, \mathbf{v}_a^{ba}, \mathbf{r}_a^{ba}, \mathbf{b}_b^{\omega}, \mathbf{b}_b^{\text{acc}}, \boldsymbol{\ell})$. A single block of the observability matrix is given by

$$
\mathbf{M}_k = \mathbf{C}(t_k)\boldsymbol{\Phi}(t_k) = \frac{\partial \mathbf{s}}{\partial \mathbf{p}} \frac{D\mathbf{p}}{D\mathcal{X}}\bigg|_{\mathbf{p}=\bar{\mathbf{C}}^{\mathsf{T}}\left(\bar{\boldsymbol{\ell}}-\bar{\mathbf{r}}\right)} \boldsymbol{\Phi}(t_k) \tag{10}
$$

Motion Type	Geometric Constraints
General 3-axis translation, 1-axis rotation	$\mathbf{C}_{b,k}\mathbf{k} = \mathbf{k}$
Constant Angular and Linear Velocity	$\mathbf{v}_k = \mathbf{v}, \quad \boldsymbol{\omega}_k = \boldsymbol{\omega}$
Constant Rotation	$\mathbf{C}_{b_k} = \mathbf{C}_b$
Local 1-Axis Translation Without Rotation	$\bar{\mathbf{C}}_{b_k} = \bar{\mathbf{C}}_b, \quad \boldsymbol{\omega}_k = \mathbf{0}, \quad \bar{\mathbf{r}}_{b_k} = f(t_k)\mathbf{k}$
Constant Velocity Without Rotation	$\mathbf{v}_k = \mathbf{v}, \quad \bar{\mathbf{C}}_{b_k} = \bar{\mathbf{C}}_b, \quad \boldsymbol{\omega}_k = \mathbf{0}$
1-Axis Rotation Without Translation	$\bar{\mathbf{C}}_{b_k}\mathbf{k} = \mathbf{k}, \quad \mathbf{r}_{b_k} = \mathbf{0}, \quad \mathbf{v}_k = \mathbf{0}$
One-Axis Angular Velocity Motion Without Translation	$\boldsymbol{\omega}_k = \boldsymbol{\omega}, \quad \mathbf{r}_{b_k} = \mathbf{0}, \quad \mathbf{v}_k = \mathbf{0}$
No Motion	$\mathbf{C}_{b_k} = \mathbf{C}_b, \quad \mathbf{v}_k = \mathbf{0}, \quad \mathbf{r}_b = \mathbf{0}, \quad \boldsymbol{\omega}_k = \mathbf{0}$

Table 1: Geometric constraints for different motion types

Denoting $\bar{\mathbf{p}} = \bar{\mathbf{C}}^{\mathsf{T}}\left(\bar{\boldsymbol{\ell}} - \bar{\mathbf{r}}\right)$, the latter right-hand product becomes

$$\left.\frac{D\mathbf{p}}{D\mathcal{X}}\right|_{\mathbf{p}=\bar{\mathbf{p}}} \boldsymbol{\Phi}(t_k) = \begin{bmatrix}\bar{\mathbf{C}}_k^{\mathsf{T}}\bar{\boldsymbol{\ell}}^{\times} & \mathbf{0} & -\bar{\mathbf{C}}_k^{\mathsf{T}} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{C}}_k^{\mathsf{T}}\end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & -\int_0^{t_k}\bar{\mathbf{C}}(t)\mathrm{d}t & \mathbf{0} & \mathbf{0} \\ t_k\mathbf{g}^{\times} & \mathbf{1} & \mathbf{0} & \boldsymbol{\Phi}_{2,4} & -\int_0^{t_k}\bar{\mathbf{C}}(t)\mathrm{d}t & \mathbf{0} \\ \frac{1}{2}t_k^2\mathbf{g}^{\times} & t_k\mathbf{1} & \mathbf{1} & \boldsymbol{\Phi}_{3,4} & \boldsymbol{\Phi}_{3,5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{11}$$

$$= \begin{bmatrix}\bar{\mathbf{C}}_k^{\mathsf{T}}\bar{\boldsymbol{\ell}}^{\times} - \frac{1}{2}t_k^2\bar{\mathbf{C}}_k^{\mathsf{T}}\mathbf{g}^{\times} & -t_k\bar{\mathbf{C}}_k^{\mathsf{T}} & -\bar{\mathbf{C}}_k^{\mathsf{T}} & -\bar{\mathbf{C}}_k^{\mathsf{T}}\bar{\boldsymbol{\ell}}^{\times}\int_0^{t_k}\bar{\mathbf{C}}(t)\mathrm{d}t - \bar{\mathbf{C}}_k^{\mathsf{T}}\boldsymbol{\Phi}_{34} & -\bar{\mathbf{C}}_k^{\mathsf{T}}\boldsymbol{\Phi}_{35} & \bar{\mathbf{C}}_k^{\mathsf{T}}\end{bmatrix} \tag{12}$$

$$= \bar{\mathbf{C}}_k^{\mathsf{T}}\begin{bmatrix}\bar{\boldsymbol{\ell}}^{\times} - \frac{1}{2}t_k^2\mathbf{g}^{\times} & -t_k\mathbf{1} & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^{\times}\int_0^{t_k}\bar{\mathbf{C}}(t)\mathrm{d}t - \boldsymbol{\Phi}_{34} & -\boldsymbol{\Phi}_{35} & \mathbf{1}\end{bmatrix} \tag{13}$$

with

$$\boldsymbol{\Phi}_{3,4} = -\mathbf{g}^{\times}\int_0^t\int_0^{\theta}\int_0^{\tau}\bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau\mathrm{d}\theta - \int_0^t\int_0^{\theta}\bar{\mathbf{v}}^{\times}(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}\tau\mathrm{d}\theta + \int_0^t\bar{\mathbf{r}}^{\times}(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}t \tag{14}$$

$$\boldsymbol{\Phi}_{3,5} = -\int_0^t\int_0^{\tau}\bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau. \tag{15}$$

The position unobservability nullspace is spanned by the columns of

$$\mathbf{N}_{\text{pos}} = \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}^{\mathsf{T}} \tag{16}$$

The yaw nullspace is non-trivial. One has to realize it involves rotation about the gravity direction, as well as moving the landmark (the first and last block columns of the observability matrix columns, respectively). Dropping all but the relevant columns for brevity, the requirement is

$$\begin{bmatrix}\bar{\boldsymbol{\ell}}^{\times} - \frac{1}{2}t_k^2\mathbf{g}^{\times} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{n}_1 \\ \mathbf{n}_2\end{bmatrix} = \mathbf{0}. \tag{17}$$

Setting $\mathbf{n}_1 = \mathbf{g}$, (by assumption)

$$\left(\bar{\boldsymbol{\ell}}^{\times} - \frac{1}{2}t_k^2\mathbf{g}^{\times}\right)\mathbf{g} + \mathbf{n}_2 = \mathbf{0} \tag{18}$$

immediately implies $\mathbf{n}_2 = -\bar{\boldsymbol{\ell}}^\times \mathbf{g}$. Therefore the yaw unobservable direction is given by

$$\mathbf{N}_{\text{yaw}} = \begin{bmatrix} \mathbf{g}^\mathsf{T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (-\bar{\boldsymbol{\ell}}^\times \mathbf{g})^\mathsf{T} \end{bmatrix}^\mathsf{T}. \tag{19}$$

3.2.1 Constant Acceleration

The following insights are required,

- The fact that the bodyframe acceleration is constant means that it may be used in the nullspace, along with the feature point $\mathbf{f}$.
- The relative position of the feature point $\mathbf{p} = \bar{\mathbf{C}}^\mathsf{T}(\boldsymbol{\ell} - \mathbf{r})$ lies in the nullspace of the projection model Jacobian,

$$\frac{\partial \mathbf{s}(\mathbf{p})}{\partial \mathbf{p}} \mathbf{p} = \mathbf{0}. \tag{20}$$

Furthermore, if bodyframe acceleration $\mathbf{a}_b^{ba}$ is constant, the following quantity (which is $\boldsymbol{\Phi}_{35}$ up to a sign), may be simplified as

$$\begin{aligned}
\left(\int_0^t \int_0^\tau \bar{\mathbf{C}}(s) \mathrm{d}s \mathrm{d}\tau \right) \mathbf{a}_b^{ba} &= \int_0^t \int_0^\tau \bar{\mathbf{C}}_{ab}(s) \mathbf{a}_b^{ba} \mathrm{d}s \mathrm{d}\tau & (21)\\
&= \int_0^t \int_0^\tau \mathbf{a}_a^{ba} \mathrm{d}s \mathrm{d}\tau & (22)\\
&= \int_0^t \mathbf{v}_a(\tau) - \mathbf{v}_a(0) \mathrm{d}\tau & (23)\\
&= \mathbf{r}_a(t) - \mathbf{r}_a(0) - t\mathbf{v}_a(0). & (24)
\end{aligned}$$

Then,

$$\mathbf{M}_k = \frac{\partial \mathbf{s}(\mathbf{p})}{\partial \mathbf{p}} \bar{\mathbf{C}}_k^\mathsf{T} \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2} t_k^2 \mathbf{g}^\times & -t_k \mathbf{1} & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t) \mathrm{d}t - \boldsymbol{\Phi}_{34} & -\boldsymbol{\Phi}_{35} & \mathbf{1} \end{bmatrix} \tag{25}$$

It is now required $\mathbf{M}_k \mathbf{n} = \mathbf{0}$. We know that, for direct relative landmark position measurements, the only unobservable directions are the standard four. Therefore, the last unobservable direction has to be obtained by considering the result of $\bar{\mathbf{C}}_k^\mathsf{T} \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2} t_k^2 \mathbf{g}^\times & -t_k \mathbf{1} & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t) \mathrm{d}t - \boldsymbol{\Phi}_{34} & -\boldsymbol{\Phi}_{35} & \mathbf{1} \end{bmatrix}$ and requiring it to be in the nullspace of the projection model Jacobian $\frac{\partial \mathbf{s}(\mathbf{p})}{\partial \mathbf{p}}$. This nullspace is spanned by the relative landmark position $\bar{\mathbf{C}}^\mathsf{T}(\boldsymbol{\ell}-\mathbf{r})$. Therefore, the nullspace vector must satisfy

$$\bar{\mathbf{C}}_k^\mathsf{T}(\boldsymbol{\ell}-\mathbf{r}_k) = \bar{\mathbf{C}}_k^\mathsf{T} \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t)\mathrm{d}t - \boldsymbol{\Phi}_{34} & \int_0^t \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau & \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_5 \\ \mathbf{n}_6 \end{bmatrix}, \tag{26}$$

$$(\boldsymbol{\ell}-\mathbf{r}_k) = \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t)\mathrm{d}t - \boldsymbol{\Phi}_{34} & \int_0^t \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_5 \\ \mathbf{n}_6 \end{bmatrix}. \tag{27}$$

Let us set $\mathbf{n}_5 = -\mathbf{a}_b$, assumed constant. This will allow to obtain $\mathbf{r}_k$ on the RHS. Then

$$(\boldsymbol{\ell}-\mathbf{r}_k) = \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t)\mathrm{d}t - \boldsymbol{\Phi}_{34} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_6 \end{bmatrix} - \mathbf{r}_k + \mathbf{r}_0 + t_k\mathbf{v}_a(0). \tag{28}$$

Let $\mathbf{n}_2 = \mathbf{v}_a(0)$. Then

$$(\boldsymbol{\ell}-\mathbf{r}_k) = \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t)\mathrm{d}t - \boldsymbol{\Phi}_{34} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_6 \end{bmatrix} - \mathbf{r}_k + \mathbf{r}_0. \tag{29}$$

Then setting $\mathbf{n}_6 = \boldsymbol{\ell}$ and $\mathbf{n}_3 = \mathbf{r}_0$ yields equality. Therefore the unobservable direction is

$$\mathbf{n} = \begin{bmatrix} \mathbf{n}_1^\mathsf{T} & \mathbf{n}_2^\mathsf{T} & \mathbf{n}_3^\mathsf{T} & \mathbf{n}_4^\mathsf{T} & \mathbf{n}_5^\mathsf{T} & \mathbf{n}_6^\mathsf{T} \end{bmatrix}^\mathsf{T} \tag{30}$$

$$= \begin{bmatrix} \mathbf{0}^\mathsf{T} & \mathbf{v}_a(0)^\mathsf{T} & \mathbf{r}_a(0)^\mathsf{T} & \mathbf{0}^\mathsf{T} & -\mathbf{a}_b^\mathsf{T} & \boldsymbol{\ell}^\mathsf{T} \end{bmatrix}^\mathsf{T}, \tag{31}$$

where the state ordering is $\mathcal{X} = (\mathcal{X}_{\text{IMU}}, \boldsymbol{\ell}) = (\mathbf{C}_{ab}, \mathbf{v}_a^{ba}, \mathbf{r}_a^{ba}, \mathbf{b}_b^\omega, \mathbf{b}_b^{\text{acc}}, \boldsymbol{\ell})$. This is the same as in VINS on Wheels.

Scale Unobservable Direction Physical Interpretation

If we do not know the physical interpretation, how do we figure it out? The fact that this is an unobservable direction, means that for a nominal state $\bar{\mathcal{X}}$, the state $\bar{\mathcal{X}} \oplus (\alpha\delta\boldsymbol{\xi}_\mathsf{n})$, where $\delta\boldsymbol{\xi}_\mathsf{n} \in$

span($\mathbf{N}$), and α some constant, yields the same measurements. Therefore,

$$\mathcal{X} = \bar{\mathcal{X}} \oplus \alpha\delta\boldsymbol{\xi}_{\mathrm{n}} = \begin{pmatrix} \mathbf{C}_{ab} \\ \mathbf{v}_a^{ba} \\ \mathbf{r}_a^{ba} \\ \mathbf{b}_b^{\omega} \\ \mathbf{b}_b^{\mathrm{acc}} \\ \boldsymbol{\ell} \end{pmatrix} \oplus \alpha \begin{bmatrix} \mathbf{0} \\ \mathbf{v}_a(0) \\ \mathbf{r}_a(0) \\ \mathbf{0} \\ -\mathbf{a}_b^{ba} \\ \boldsymbol{\ell} \end{bmatrix} = \begin{pmatrix} \delta\mathbf{C}\mathbf{C}_{ab} \\ \delta\mathbf{v} + \delta\mathbf{C}\bar{\mathbf{v}}_a^{ba} \\ \delta\mathbf{r} + \delta\mathbf{C}\bar{\mathbf{r}}_a^{ba} \\ \mathbf{b}_b^{\omega} + \delta\mathbf{b}^{\omega} \\ \mathbf{b}_b^{\mathrm{acc}} + \delta\mathbf{b}^{\mathrm{acc}} \\ \boldsymbol{\ell} + \delta\boldsymbol{\ell} \end{pmatrix} \tag{32}$$

$$= \begin{pmatrix} \mathbf{C}_{ab} \\ \delta\boldsymbol{\xi}^{\mathrm{v}} + \bar{\mathbf{v}}_a^{ba} \\ \delta\boldsymbol{\xi}^{\mathrm{r}} + \bar{\mathbf{r}}_a^{ba} \\ \mathbf{b}_b^{\omega} + \delta\mathbf{b}^{\omega} \\ \mathbf{b}_b^{\mathrm{acc}} + \delta\mathbf{b}^{\mathrm{acc}} \\ \boldsymbol{\ell} + \delta\boldsymbol{\ell} \end{pmatrix} \tag{33}$$

$$= \begin{pmatrix} \mathbf{C}_{ab} \\ \alpha\bar{\mathbf{v}}_0 + \bar{\mathbf{v}}_a^{ba} \\ \alpha\bar{\mathbf{r}}_0 + \bar{\mathbf{r}}_a^{ba} \\ \mathbf{b}_b^{\omega} \\ \mathbf{b}_b^{\mathrm{acc}} - \alpha\mathbf{a}_b^{ba} \\ \boldsymbol{\ell} + \alpha\boldsymbol{\ell} \end{pmatrix} \tag{34}$$

$$= \begin{pmatrix} \mathbf{C}_{ab} \\ \bar{\mathbf{v}}_a^{ba} \\ \bar{\mathbf{r}}_a^{ba} \\ \mathbf{b}_b^{\omega} \\ \mathbf{b}_b^{\mathrm{acc}} \\ \boldsymbol{\ell} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \alpha\bar{\mathbf{v}}_0 \\ \alpha\bar{\mathbf{r}}_0 \\ \mathbf{0} \\ -\alpha\mathbf{a}_b^{ba} \\ \alpha\boldsymbol{\ell} \end{pmatrix} \tag{35}$$

where matrix notation was abused to stack the substates vertically. Since this observability argument holds for any time t_k instead of t_0, one can now substitute $\bar{\mathcal{X}}$ for $\bar{\mathcal{X}}_0$

$$\mathcal{X} = \begin{pmatrix} \mathbf{C}_{ab} \\ \bar{\mathbf{v}}_a^{ba} \\ \bar{\mathbf{r}}_a^{ba} \\ \mathbf{b}_b^{\omega} \\ \mathbf{b}_b^{\mathrm{acc}} \\ \boldsymbol{\ell} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \alpha\bar{\mathbf{v}}_0 \\ \alpha\bar{\mathbf{r}}_0 \\ \mathbf{0} \\ -\alpha\mathbf{a}_b^{ba} \\ \alpha\boldsymbol{\ell} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{ab} \\ \bar{\mathbf{v}}_a^{ba} \\ \bar{\mathbf{r}}_a^{ba} \\ \mathbf{b}_b^{\omega} \\ \mathbf{b}_b^{\mathrm{acc}} \\ \boldsymbol{\ell} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \alpha\bar{\mathbf{v}}_a^{ba} \\ \alpha\bar{\mathbf{r}}_a^{ba} \\ \mathbf{0} \\ -\alpha\mathbf{a}_b^{ba} \\ \alpha\boldsymbol{\ell} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \alpha\bar{\mathbf{v}}_a^{ba} \\ \alpha\bar{\mathbf{r}}_a^{ba} \\ \mathbf{0} \\ \mathbf{b}_b^{\mathrm{acc}} - (\alpha-1)\mathbf{a}_b^{ba} \\ \alpha\boldsymbol{\ell} \end{pmatrix}, \tag{36}$$

where the substitution $\alpha \leftarrow \alpha - 1$ was made in the last equality, since α can be any scalar. One can then guess the scale transformation, where the robot position and landmarks are all scaled by α. It is clear that the position, landmarks, and velocity will be scaled by α. The accelerometer will measure

$$\mathbf{u}^{\mathrm{acc}} = \mathbf{C}_{ab}^{\mathsf{T}}(\alpha\mathbf{a}_a^{ba} - \mathbf{g}) + \mathbf{b}^{\mathrm{acc}} \tag{37}$$

$$= \mathbf{C}_{ab}^{\mathsf{T}}(\mathbf{a}_a^{ba} - \mathbf{g}) + (\alpha-1)\mathbf{a}_b^{ba} + \mathbf{b}^{\mathrm{acc}} \tag{38}$$

$$= \mathbf{C}_{ab}^{\mathsf{T}}(\mathbf{a}_a^{ba} - \mathbf{g}) + (\alpha-1)\mathbf{a}_b^{ba} + \mathbf{b}^{\mathrm{acc}}. \tag{39}$$

If we set a new bias to $\tilde{\mathbf{b}}^{\text{acc}} = \mathbf{b}^{\text{acc}} - (\alpha - 1)\mathbf{a}_b^{ba}$, we recover the same accelerometer measurement. This is the same as the direction that was found through the non-observable directions.

3.2.2 Constant Rotation

Same procedure, a single block of the observability matrix is given by

$$\mathbf{M}_k = \mathbf{C}(t_k)\boldsymbol{\Phi}(t_k) = \frac{\partial \mathbf{s}}{\partial \mathbf{p}} \frac{D\mathbf{p}}{D\mathcal{X}}\bigg|_{\mathbf{p}=\bar{\mathbf{C}}^\mathsf{T}\left(\bar{\boldsymbol{\ell}}-\bar{\mathbf{r}}\right)} \boldsymbol{\Phi}(t_k) \tag{40}$$

Denoting $\bar{\mathbf{p}} = \bar{\mathbf{C}}^\mathsf{T}\left(\bar{\boldsymbol{\ell}} - \bar{\mathbf{r}}\right)$, the latter right-hand product becomes

$$\frac{D\mathbf{p}}{D\mathcal{X}}\bigg|_{\mathbf{p}=\bar{\mathbf{p}}} \boldsymbol{\Phi}(t_k) = \bar{\mathbf{C}}_k^\mathsf{T} \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t)\mathrm{d}t - \boldsymbol{\Phi}_{34} & -\boldsymbol{\Phi}_{35} & \mathbf{1} \end{bmatrix} \tag{41}$$

with

$$\boldsymbol{\Phi}_{3,4} = -\mathbf{g}^\times \int_0^t \int_0^\theta \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau\mathrm{d}\theta - \int_0^t \int_0^\theta \bar{\mathbf{v}}^\times(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}\tau\mathrm{d}\theta + \int_0^t \bar{\mathbf{r}}^\times(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}t \tag{42}$$

$$\boldsymbol{\Phi}_{3,5} = -\int_0^t \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau. \tag{43}$$

For constant rotation,

$$\begin{aligned} \boldsymbol{\Phi}_{3,4} &= -\mathbf{g}^\times \int_0^t \int_0^\theta \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau\mathrm{d}\theta - \int_0^t \int_0^\theta \bar{\mathbf{v}}^\times(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}\tau\mathrm{d}\theta + \int_0^t \bar{\mathbf{r}}^\times(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}t & (44)\\ &= -\mathbf{g}^\times\bar{\mathbf{C}} \int_0^t \int_0^\theta \int_0^\tau \mathrm{d}s\mathrm{d}\tau\mathrm{d}\theta - \int_0^t (\bar{\mathbf{r}}(\theta) - \bar{\mathbf{r}}_0)^\times \mathrm{d}\theta\bar{\mathbf{C}} + \int_0^t \bar{\mathbf{r}}^\times(\tau)\bar{\mathbf{C}}(\tau)\mathrm{d}t & (45)\\ &= -\frac{t^3}{6}\mathbf{g}^\times\bar{\mathbf{C}} + t\bar{\mathbf{r}}_0^\times\bar{\mathbf{C}}, & (46)\\ \boldsymbol{\Phi}_{3,5} &= -\int_0^t \int_0^\tau \bar{\mathbf{C}}(s)\mathrm{d}s\mathrm{d}\tau = -\frac{t^2}{2}\bar{\mathbf{C}}. & (47) \end{aligned}$$

For constant orientation, the fourth block of the $\frac{D\mathbf{p}}{D\mathcal{X}}$ matrix becomes

$$\begin{aligned} -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t)\mathrm{d}t - \boldsymbol{\Phi}_{34} &= -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t)\mathrm{d}t - \left(-\frac{t^3}{6}\mathbf{g}^\times\bar{\mathbf{C}} + t\bar{\mathbf{r}}_0^\times\bar{\mathbf{C}}\right) & (48)\\ &= -t_k\bar{\boldsymbol{\ell}}^\times\bar{\mathbf{C}} + \frac{t^3}{6}\mathbf{g}^\times\bar{\mathbf{C}} - t\bar{\mathbf{r}}_0^\times\bar{\mathbf{C}}. & (49) \end{aligned}$$

Therefore, once more considering the latter right-hand product with the constant orientation assumption,

$$\begin{aligned} \frac{D\mathbf{p}}{D\mathcal{X}}\bigg|_{\mathbf{p}=\bar{\mathbf{p}}} &= \bar{\mathbf{C}}_k^\mathsf{T} \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -\bar{\boldsymbol{\ell}}^\times \int_0^{t_k} \bar{\mathbf{C}}(t)\mathrm{d}t - \boldsymbol{\Phi}_{34} & -\boldsymbol{\Phi}_{35} & \mathbf{1} \end{bmatrix} & (50)\\ &= \bar{\mathbf{C}}_k^\mathsf{T} \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -t_k\bar{\boldsymbol{\ell}}^\times\bar{\mathbf{C}} + \frac{t^3}{6}\mathbf{g}^\times\bar{\mathbf{C}} - t\bar{\mathbf{r}}_0^\times\bar{\mathbf{C}} & -(-\frac{t^2}{2}\bar{\mathbf{C}}) & \mathbf{1} \end{bmatrix} & (51)\\ &= \bar{\mathbf{C}}_k^\mathsf{T} \begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -t_k\bar{\boldsymbol{\ell}}^\times\bar{\mathbf{C}} + \frac{t^3}{6}\mathbf{g}^\times\bar{\mathbf{C}} - t\bar{\mathbf{r}}_0^\times\bar{\mathbf{C}} & \frac{t^2}{2}\bar{\mathbf{C}} & \mathbf{1} \end{bmatrix}. & (52) \end{aligned}$$

Then considering the nullspace of the right-hand matrix, s

$$\begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -t_k\bar{\boldsymbol{\ell}}^\times\bar{\mathbf{C}} + \frac{t^3}{6}\mathbf{g}^\times\bar{\mathbf{C}} - t\bar{\mathbf{r}}_0^\times\bar{\mathbf{C}} & \frac{t^2}{2}\bar{\mathbf{C}} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_5 \\ \mathbf{n}_6 \end{bmatrix}. \tag{53}$$

Consider the answer from VINS on wheels, where

$$\mathbf{N} = \begin{bmatrix} \bar{\mathbf{C}} \\ -\mathbf{v}_0^\times \\ -\mathbf{r}_0^\times \\ \mathbf{0}^{3\times 3} \\ \mathbf{g}^\times\bar{\mathbf{C}}^\mathsf{T}\mathbf{r}_0^\times \\ -\boldsymbol{\ell}^\times \end{bmatrix} \tag{54}$$

I don't think this works for us. Perhaps its the error definition. Let's try to figure something out.

$$\begin{bmatrix} \bar{\boldsymbol{\ell}}^\times - \frac{1}{2}t_k^2\mathbf{g}^\times & -t_k\mathbf{1} & -\mathbf{1} & -t_k\bar{\boldsymbol{\ell}}^\times\bar{\mathbf{C}} + \frac{t^3}{6}\mathbf{g}^\times\bar{\mathbf{C}} - t\bar{\mathbf{r}}_0^\times\bar{\mathbf{C}} & \frac{t^2}{2}\bar{\mathbf{C}} & \mathbf{1} \end{bmatrix} \mathbf{N} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \bar{\mathbf{C}}^\mathsf{T}\mathbf{g}^\times \\ -\boldsymbol{\ell}^\times \end{bmatrix}. \tag{55}$$

Constant Rotation Unobservability Physical Interpretation

We consider the state $\mathcal{X}$ perturbed by a linear combination of the columns of $\mathbf{N}$, $\mathbf{N}\delta\boldsymbol{\xi}$

$$\mathcal{X} = \bar{\mathcal{X}} \oplus \alpha\delta\boldsymbol{\xi}_\text{n} = \begin{pmatrix} \mathbf{C}_{ab} \\ \mathbf{v}_a^{ba} \\ \mathbf{r}_a^{ba} \\ \mathbf{b}_b^{\omega} \\ \mathbf{b}_b^{\text{acc}} \\ \boldsymbol{\ell} \end{pmatrix} \oplus \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \bar{\mathbf{C}}^\mathsf{T}\mathbf{g}^\times \\ -\boldsymbol{\ell}^\times \end{bmatrix} \delta\boldsymbol{\xi} = \begin{pmatrix} \bar{\mathbf{C}}_{ab} \\ \bar{\mathbf{v}}_a^{ba} \\ \bar{\mathbf{r}}_a^{ba} \\ \bar{\mathbf{b}}_b^{\omega} \\ \bar{\mathbf{b}}_b^{\text{acc}} \\ \bar{\boldsymbol{\ell}} \end{pmatrix} \oplus \begin{bmatrix} \delta\boldsymbol{\xi} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \bar{\mathbf{C}}^\mathsf{T}\mathbf{g}^\times\delta\boldsymbol{\xi} \\ -\bar{\boldsymbol{\ell}}^\times\delta\boldsymbol{\xi} \end{bmatrix} \tag{56}$$

$$= \begin{pmatrix} \operatorname{Exp}(\delta\boldsymbol{\xi})\mathbf{C}_{ab} \\ \bar{\mathbf{v}}_a^{ba} \\ \bar{\mathbf{r}}_a^{ba} \\ \bar{\mathbf{b}}_b^{\omega} \\ \bar{\mathbf{b}}_b^{\text{acc}} + \bar{\mathbf{C}}^\mathsf{T}\mathbf{g}^\times\delta\boldsymbol{\xi} \\ \bar{\boldsymbol{\ell}} - \bar{\boldsymbol{\ell}}^\times\delta\boldsymbol{\xi} \end{pmatrix}. \tag{57}$$

If we rotate the robot as $\mathbf{C}_{ab} \leftarrow (\mathbf{1} + \delta\boldsymbol{\xi}^\times)\,\bar{\mathbf{C}}_{ab}$, and consider the accelerometer measurements, we get

$$
\begin{aligned}
\mathbf{u}^{\text{acc}} &= \left(\left(\mathbf{1} + \delta\boldsymbol{\xi}^\times\right)\bar{\mathbf{C}}_{ab}\right)^\mathsf{T}\left(\mathbf{a}_a^{ba} - \mathbf{g}\right) + \mathbf{b}^{\text{acc}} && (58)\\
&= \bar{\mathbf{C}}_{ab}^\mathsf{T}\left(\mathbf{1} - \delta\boldsymbol{\xi}^\times\right)\left(\mathbf{a}_a^{ba} - \mathbf{g}\right) + \mathbf{b}^{\text{acc}} && (59)\\
&= \bar{\mathbf{C}}_{ab}^\mathsf{T}\left(\mathbf{1} - \delta\boldsymbol{\xi}^\times\right)\mathbf{a}_a^{ba} + \bar{\mathbf{C}}_{ab}^\mathsf{T}\mathbf{g} - \bar{\mathbf{C}}_{ab}^\mathsf{T}\delta\boldsymbol{\xi}^\times\mathbf{g} + \mathbf{b}^{\text{acc}} && (60)\\
& && (61)
\end{aligned}
$$

TODO Iron this out, this is half baked for now.

3.3 Degenerate Multisensor: SE(3) Odometry + Global Pose

The first notation is used when kinematics are important, the second when linearizing and using error Jacobians. $\mathcal{X} = \left(\mathbf{C}_{ab}, \mathbf{r}_a^{ba}, \mathbf{C}_{bc}, \mathbf{r}_b^{cb}, \tau_{bc}.\right)$, and $\mathcal{X} = (\mathbf{C}_b, \mathbf{r}_b, \mathbf{C}_c, \mathbf{r}_c, \tau)$. The k'th block of the observability matrix is given by

$$
\begin{aligned}
\mathbf{M}_k &= \mathbf{C}_k\boldsymbol{\Phi}_k && (62)\\
&= \mathbf{C}_k && (63)\\
&= \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^\times + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^\times\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^\times & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^\times\boldsymbol{\omega}_k \end{bmatrix}. && (64)
\end{aligned}
$$

The quantities $\bar{\mathbf{C}}_b, \bar{\mathbf{r}}_b$ are in general time-varying, evaluated at time t_k. The quantity $\bar{\mathbf{r}}_c$ is time-invariant.

3.3.1 General 3-axis translation, 1-axis rotation

For a 1-axis rotation about a vector $\mathbf{k}$, the geometric constraint is

$$
\mathbf{C}_{b,k}\mathbf{k} = \mathbf{k}. \tag{65}
$$

We can try the following nullspace vector

$$
\mathbf{M}_k\mathbf{n} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^\times + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^\times\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^\times & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^\times\boldsymbol{\omega}_k \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ -\mathbf{k} \\ \mathbf{0} \\ \mathbf{k} \\ \mathbf{0} \end{bmatrix}. \tag{66}
$$

The physical transformation is given by

$$
\begin{pmatrix} \mathbf{C}_{ab_k} \\ \mathbf{r}_a^{b_k a} \\ \mathbf{C}_{bc} \\ \mathbf{r}_b^{cb} \\ \tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \begin{bmatrix} \delta\boldsymbol{\xi}_b^\phi \\ \delta\boldsymbol{\xi}_b^\mathrm{r} \\ \delta\boldsymbol{\xi}_c^\phi \\ \delta\boldsymbol{\xi}_c^\mathrm{r} \\ \delta\tau_{cb} \end{bmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \alpha \begin{bmatrix} \mathbf{0} \\ -\mathbf{k} \\ \mathbf{0} \\ \mathbf{k} \\ 0 \end{bmatrix} = \begin{pmatrix} \delta\mathbf{C}_b\bar{\mathbf{C}}_{ab_k} \\ \delta\mathbf{r}_b + \delta\mathbf{C}_b\bar{\mathbf{r}}_a^{b_k a} \\ \delta\mathbf{C}_c\bar{\mathbf{C}}_{bc} \\ \delta\mathbf{r}_c + \delta\mathbf{C}_c\bar{\mathbf{r}}_b^{cb} \\ \delta\tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ -\alpha\mathbf{k} + \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \alpha\mathbf{k} + \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix}. \tag{67}
$$

For planar motion, with $\mathbf{k} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\mathsf{T}$, the z-component of the extrinsics is unobservable (can vary together with position in z direction).

3.3.2 Constant Angular and Linear Velocity

For a 1-axis rotation about a vector $\mathbf{k}$, the geometric constraint is

$$\mathbf{C}_{b,k}\mathbf{k} = \mathbf{k}. \tag{68}$$

Furthermore, angular and linear velocity is constant,

$$\mathbf{v}_k = \mathbf{v}, \quad \boldsymbol{\omega}_k = \boldsymbol{\omega}. \tag{69}$$

$$\mathbf{M}_k\mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix}\mathbf{n}. \tag{70}$$

We can try the nullspace vector,

$$\begin{aligned} \mathbf{M}_k\mathbf{n}_k &= \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \\ \boldsymbol{\omega} \\ \mathbf{0} \\ 1 \end{bmatrix} & (71) \\ &= \begin{bmatrix} \bar{\mathbf{C}}_{b_k}\boldsymbol{\omega} - \bar{\mathbf{C}}_{b_k}\boldsymbol{\omega} \\ \mathbf{v} - \bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times}\boldsymbol{\omega} - \mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} = \mathbf{0}. & (72) \end{aligned}$$

The physical transformation is given by

$$\begin{pmatrix} \mathbf{C}_{ab_k} \\ \mathbf{r}_a^{b_k a} \\ \mathbf{C}_{bc} \\ \mathbf{r}_b^{cb} \\ \tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \alpha \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \\ \boldsymbol{\omega} \\ \mathbf{0} \\ 1 \end{bmatrix} = \begin{pmatrix} \delta\mathbf{C}_b\bar{\mathbf{C}}_{ab_k} \\ \delta\mathbf{r}_b + \delta\mathbf{C}_b\bar{\mathbf{r}}_a^{b_k a} \\ \delta\mathbf{C}_c\bar{\mathbf{C}}_{bc} \\ \delta\mathbf{r}_c + \delta\mathbf{C}_c\bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} + \delta\tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \alpha\mathbf{v} + \bar{\mathbf{r}}_a^{b_k a} \\ \mathrm{Exp}(\alpha\boldsymbol{\omega})\bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \alpha + \bar{\tau}_{cb} \end{pmatrix}. \tag{73}$$

No clear interpretation is available here. However, we are able to change the odometry sensor position, extrinsic rotation, and time offset together.

3.3.3 Constant Rotation

For constant rotation,

$$\mathbf{C}_{b_k} = \mathbf{C}_b. \tag{74}$$

Can try,

$$\mathbf{M}_k\mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{C}}_{b_k} \\ \mathbf{0} \\ -\mathbf{1} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}. \tag{75}$$

Physical interpretation is given by

$$\begin{pmatrix} \mathbf{C}_{ab_k} \\ \mathbf{r}_a^{b_k a} \\ \mathbf{C}_{bc} \\ \mathbf{r}_b^{cb} \\ \tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{C}}_{b_k} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} \boldsymbol{\alpha} = \begin{pmatrix} \delta\mathbf{C}_b \bar{\mathbf{C}}_{ab_k} \\ \delta\mathbf{r}_b + \delta\mathbf{C}_b \bar{\mathbf{r}}_a^{b_k a} \\ \delta\mathbf{C}_c \bar{\mathbf{C}}_{bc} \\ \delta\mathbf{r}_c + \delta\mathbf{C}_c \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} + \delta\tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{C}}_{ab_k}\boldsymbol{\alpha} + \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ -\boldsymbol{\alpha} + \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix}. \tag{76}$$

This means the robot position is unobservable, varying together with the position extrinsic. The $\bar{\mathbf{C}}_{ab_k}$ takes care of the reference frame change.

3.3.4 Local 1-Axis Translation Without Rotation

The geometric constraint is

$$\bar{\mathbf{C}}_{b_k} = \bar{\mathbf{C}}_b, \quad \boldsymbol{\omega}_k = \mathbf{0}, \quad \bar{\mathbf{r}}_{b_k} = f(t_k)\mathbf{k}. \tag{77}$$

Can try,

$$\mathbf{M}_k \mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \mathbf{k} \\ \mathbf{0} \\ \mathbf{n}_3 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{78}$$

$$= \begin{bmatrix} \mathbf{k} + \bar{\mathbf{C}}_{b_k}\mathbf{n}_3 \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right)\mathbf{k} - \bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times}\mathbf{n}_3 \end{bmatrix} \tag{79}$$

$$= \begin{bmatrix} \mathbf{k} + \bar{\mathbf{C}}_{b_k}\mathbf{n}_3 \\ -(\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\mathbf{k} - \bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times}\mathbf{n}_3 \end{bmatrix}, \tag{80}$$

setting $\mathbf{n}_3 = -\bar{\mathbf{C}}_{b_k}^{\mathsf{T}}\mathbf{k}$ yields

$$\mathbf{M}_k \mathbf{n}_k = \begin{bmatrix} \mathbf{k} - \bar{\mathbf{C}}_{b_k}\bar{\mathbf{C}}_{b_k}\mathbf{k} \\ -(\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\mathbf{k} + \bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times}\bar{\mathbf{C}}_{b_k}^{\mathsf{T}}\mathbf{k} \end{bmatrix} = \mathbf{0}. \tag{81}$$

The unobservable direction is thus

$$\mathbf{n} = \begin{bmatrix} \mathbf{k} \\ \mathbf{0} \\ -\bar{\mathbf{C}}_{b_k}^{\mathsf{T}}\mathbf{k} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{82}$$

The physical interpretation is

$$\begin{pmatrix} \mathbf{C}_{ab_k} \\ \mathbf{r}_a^{b_k a} \\ \mathbf{C}_{bc} \\ \mathbf{r}_b^{cb} \\ \tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \alpha \begin{bmatrix} \mathbf{k} \\ \mathbf{0} \\ -\bar{\mathbf{C}}_{b_k}^{\mathsf{T}}\mathbf{k} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{pmatrix} \delta\mathbf{C}_b \bar{\mathbf{C}}_{ab_k} \\ \delta\mathbf{r}_b + \delta\mathbf{C}_b \bar{\mathbf{r}}_a^{b_k a} \\ \delta\mathbf{C}_c \bar{\mathbf{C}}_{bc} \\ \delta\mathbf{r}_c + \delta\mathbf{C}_c \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} + \delta\tau_{cb} \end{pmatrix} = \begin{pmatrix} \mathrm{Exp}(\alpha\mathbf{k})\bar{\mathbf{C}}_{ab_k} \\ \mathrm{Exp}(\alpha\mathbf{k})\bar{\mathbf{r}}_a^{b_k a} \\ \mathrm{Exp}(-\alpha\bar{\mathbf{C}}_{b_k}^{\mathsf{T}}\mathbf{k})\bar{\mathbf{C}}_{bc} \\ \mathrm{Exp}(-\alpha\bar{\mathbf{C}}_{b_k}^{\mathsf{T}}\mathbf{k})\bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \tag{83}$$

We can rotate extrinsics and body rotation in the opposite directions to recover the same measurements.

3.3.5 Constant Velocity Without Rotation

The geometric constraint is

$$\mathbf{v}_k = \mathbf{v}, \quad \bar{\mathbf{C}}_{b_k} = \bar{\mathbf{C}}_b, \quad \boldsymbol{\omega}_k = \mathbf{0} \tag{84}$$

Can try,

$$\mathbf{M}_k \mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix} \tag{85}$$

Physical interpretation is

$$\begin{pmatrix} \mathbf{C}_{ab_k} \\ \mathbf{r}_a^{b_k a} \\ \mathbf{C}_{bc} \\ \mathbf{r}_b^{cb} \\ \tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \alpha \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix} = \begin{pmatrix} \delta\mathbf{C}_b\bar{\mathbf{C}}_{ab_k} \\ \delta\mathbf{r}_b + \delta\mathbf{C}_b\bar{\mathbf{r}}_a^{b_k a} \\ \delta\mathbf{C}_c\bar{\mathbf{C}}_{bc} \\ \delta\mathbf{r}_c + \delta\mathbf{C}_c\bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} + \delta\tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \alpha\mathbf{v} + \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \alpha + \bar{\tau}_{cb} \end{pmatrix}. \tag{86}$$

Changing the robot position along the velocity allows us to change the time offset and recover the same measurements.

3.3.6 1-Axis Rotation Without Translation

The geometric constraint is

$$\bar{\mathbf{C}}_{b_k}\mathbf{k} = \mathbf{k}, \quad \mathbf{r}_{b_k} = \mathbf{0}, \quad \mathbf{v}_k = \mathbf{0}, \tag{87}$$

Can try,

$$\mathbf{M}_k \mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ -\mathbf{k} \\ \mathbf{0} \\ \mathbf{k} \\ \mathbf{0} \end{bmatrix}, \tag{88}$$

as well as

$$\mathbf{M}_k \mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} -\mathbf{k} \\ \mathbf{0} \\ \mathbf{C}_{b_k}^{\mathsf{T}}\mathbf{k} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{89}$$

such that

$$\mathbf{N} = \begin{bmatrix} \mathbf{0} & -\mathbf{k} \\ -\mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{b_k}^{\mathsf{T}}\mathbf{k} \\ \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{90}$$

Physical interpretation is given by

$$\begin{aligned}
\begin{pmatrix} \mathbf{C}_{ab_k} \\ \mathbf{r}_a^{b_k a} \\ \mathbf{C}_{bc} \\ \mathbf{r}_b^{cb} \\ \tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \mathbf{N}\boldsymbol{\alpha} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \begin{bmatrix} \mathbf{0} & -\mathbf{k} \\ -\mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{b_k}^{\mathsf{T}}\mathbf{k} \\ \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{\alpha} &= \begin{pmatrix} \delta\mathbf{C}_b \bar{\mathbf{C}}_{ab_k} \\ \delta\mathbf{r}_b + \delta\mathbf{C}_b \bar{\mathbf{r}}_a^{b_k a} \\ \delta\mathbf{C}_c \bar{\mathbf{C}}_{bc} \\ \delta\mathbf{r}_c + \delta\mathbf{C}_c \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} + \delta\tau_{cb} \end{pmatrix} \qquad (91) \\
&= \begin{pmatrix} \mathrm{Exp}(-\alpha_2\mathbf{k})\bar{\mathbf{C}}_{ab_k} \\ -\alpha_1\mathbf{k} + \mathrm{Exp}(-\alpha_2\mathbf{k})\bar{\mathbf{r}}_a^{b_k a} \\ \mathrm{Exp}(\alpha_2\mathbf{C}_{b_k}^{\mathsf{T}}\mathbf{k})\bar{\mathbf{C}}_{bc} \\ \alpha_1\mathbf{k} + \mathrm{Exp}(\alpha_2\mathbf{C}_{b_k}^{\mathsf{T}}\mathbf{k})\bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \qquad (92) \\
&= \begin{pmatrix} \mathbf{0} \\ -\alpha_1\mathbf{k} \\ \mathbf{0} \\ \alpha_1\mathbf{k} \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{Exp}(-\alpha_2\mathbf{k})\bar{\mathbf{C}}_{ab_k} \\ \mathrm{Exp}(-\alpha_2\mathbf{k})\bar{\mathbf{r}}_a^{b_k a} \\ \mathrm{Exp}(\alpha_2\mathbf{C}_{b_k}^{\mathsf{T}}\mathbf{k})\bar{\mathbf{C}}_{bc} \\ \mathrm{Exp}(\alpha_2\mathbf{C}_{b_k}^{\mathsf{T}}\mathbf{k})\bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix}. \qquad (93)
\end{aligned}$$

We can rotate extrinsics and robot pose in opposite directions, as well as shift extrinsic and robot position in opposite diretions, along the axis of movement.

3.3.7 One-Axis Angular Velocity Motion Without Translation

The geometric constraint is

$$\boldsymbol{\omega}_k = \boldsymbol{\omega}, \quad \mathbf{r}_{b_k} = \mathbf{0}, \quad \mathbf{v}_k = \mathbf{0}. \tag{94}$$

Can try,

$$\mathbf{M}_k\mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \bar{\mathbf{C}}_{b_k}\boldsymbol{\omega} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix}. \tag{95}$$

Physical interpretation,

$$\begin{pmatrix} \mathbf{C}_{ab_k} \\ \mathbf{r}_a^{b_k a} \\ \mathbf{C}_{bc} \\ \mathbf{r}_b^{cb} \\ \tau_{cb} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{C}}_{ab_k} \\ \bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} \end{pmatrix} \oplus \alpha \begin{bmatrix} \bar{\mathbf{C}}_{b_k}\boldsymbol{\omega} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix} = \begin{pmatrix} \delta\mathbf{C}_b \bar{\mathbf{C}}_{ab_k} \\ \delta\mathbf{r}_b + \delta\mathbf{C}_b \bar{\mathbf{r}}_a^{b_k a} \\ \delta\mathbf{C}_c \bar{\mathbf{C}}_{bc} \\ \delta\mathbf{r}_c + \delta\mathbf{C}_c \bar{\mathbf{r}}_b^{cb} \\ \bar{\tau}_{cb} + \delta\tau_{cb} \end{pmatrix} = \begin{pmatrix} \mathrm{Exp}(\alpha\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega})\bar{\mathbf{C}}_{ab_k} \\ \mathrm{Exp}(\alpha\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega})\bar{\mathbf{r}}_a^{b_k a} \\ \bar{\mathbf{C}}_{bc} \\ \bar{\mathbf{r}}_b^{cb} \\ \alpha + \bar{\tau}_{cb} \end{pmatrix} \tag{96}$$

3.3.8 No Motion

Geometric constraints are

$$\mathbf{C}_{b_k} = \mathbf{C}_b, \mathbf{v}_k = \mathbf{0}, \mathbf{r}_b = \mathbf{0}, \boldsymbol{\omega}_k = \mathbf{0}. \tag{97}$$

Can try

$$\mathbf{M}_k \mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix}, \tag{98}$$

as well as

$$\mathbf{M}_k \mathbf{n}_k = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \bar{\mathbf{C}}_{b_k} & \mathbf{0} & -\bar{\mathbf{C}}_{b_k}\boldsymbol{\omega}_k \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\mathbf{v}_k + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \bar{\mathbf{C}}_{b_k} \\ \mathbf{0} \\ -\mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{99}$$

$$= \begin{bmatrix} \bar{\mathbf{C}}_{b_k} - \bar{\mathbf{C}}_{b_k} \\ -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right)\bar{\mathbf{C}}_{b_k} + \bar{\mathbf{C}}_{b_k}\mathbf{r}_c^{\times} \end{bmatrix} = \mathbf{0}. \tag{100}$$

Time offset is unobservable. Furthermore, robot orientation and orientation extrinsics can be rotated opposite way to get same measurements.

3.4 Odometry + GPS

The state transition matrix is identity. Measurement model Jacobian is just the position portion of the global pose measurement model. Thus,

$$\mathbf{C}_k = \begin{bmatrix} -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\bar{\mathbf{v}}_k + \bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \tag{101}$$

and,

$$\mathbf{M}_k = \mathbf{C}_k \boldsymbol{\Phi}_k = \mathbf{C}_k = \begin{bmatrix} -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\bar{\mathbf{v}}_k + \bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix}. \tag{102}$$

Nullspace vector equation may be formed as

$$\mathbf{M}_k \mathbf{n} = \begin{bmatrix} -\left(\bar{\mathbf{r}}_{b_k}^{\times} + (\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c)^{\times}\right) & \mathbf{1} & -\bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times} & \bar{\mathbf{C}}_{b_k} & -\bar{\mathbf{v}}_k + \bar{\mathbf{C}}_{b_k}\bar{\mathbf{r}}_c^{\times}\boldsymbol{\omega}_k \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_5 \end{bmatrix} \tag{103}$$

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