IMU Kinematics, Strapdown Navigation, Preintegration

Vassili Korotkine
Department of Mechanical Engineering, McGill University
817 Sherbrooke Street West, Montreal QC H3A 0C3

September 16, 2026

This document is a living document and a record of my thought process as I explore these topics. It is not meant as an authoritative reference, and there will be missing citations and typos.

1 Angular Transport Theorem

A physical vector's derivative is different based on the reference frame used. The angular transport theorem states that

$$\underrightarrow{r}^{\bullet a} = \underrightarrow{r}^{\bullet b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{r}, \tag{1}$$

which can be rewritten

$$\underrightarrow{v}^{/a} = \underrightarrow{v}^{/b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{r}. \tag{2}$$

Typically, one would see something similar to $\underrightarrow{v}^{zw/a}$, where z and w are physical points. These points are dropped since the physical vector under consideration is an arbitrary $\underrightarrow{r}$.

Therefore, the acceleration is given by

$${\underrightarrow{v}^{/a}}^{\bullet a} = {\underrightarrow{v}^{/b}}^{\bullet a} + \left(\underrightarrow{\omega}^{ba} \times \underrightarrow{r}\right)^{\bullet a}. \tag{3}$$

The first term is manipulated using angular transport theorem,

$$\begin{aligned} {\underrightarrow{v}^{/b}}^{\bullet a} &= {\underrightarrow{v}^{/b}}^{\bullet b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/b} && (4) \\ &= \underrightarrow{a}^{/b/b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/b}. && (5) \end{aligned}$$

Second term is manipulated using product rule,

$$\left(\underrightarrow{\omega}^{ba} \times \underrightarrow{r}\right)^{\bullet a} = \underrightarrow{\alpha}^{ba/a} \times \underrightarrow{r} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/a} \tag{6}$$

$$= \underrightarrow{\alpha}^{ba/a} \times \underrightarrow{r} + \underrightarrow{\omega}^{ba} \times \left(\underrightarrow{v}^{/b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{r}\right) \tag{7}$$

$$= \underrightarrow{\alpha}^{ba/a} \times \underrightarrow{r} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/b} + \underrightarrow{\omega}^{ba} \times \left(\underrightarrow{\omega}^{ba} \times \underrightarrow{r}\right) .. \tag{8}$$

Adding (5) and (8) yields the expression for $\underrightarrow{a}^{/a/a} = \underrightarrow{v}^{/a^{\bullet a}}$,

$$\underrightarrow{a}^{/a/a} = \underrightarrow{a}^{/b/b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/b} + \underrightarrow{\alpha}^{ba/a} \times \underrightarrow{r} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/b} + \underrightarrow{\omega}^{ba} \times \left(\underrightarrow{\omega}^{ba} \times \underrightarrow{r}\right) \tag{9}$$

$$= \underrightarrow{a}^{/b/b} + 2\underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/b} + \underrightarrow{\alpha}^{ba/a} \times \underrightarrow{r} + \underrightarrow{\omega}^{ba} \times \left(\underrightarrow{\omega}^{ba} \times \underrightarrow{r}\right), \tag{10}$$

which links accelerations with respect to different frames. This difference in accelerations with respect to different reference frames is, essentially, the Coriolis effect. The Wikipedia article names for the terms (as forces) are “Coriolis force” for $2\underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/b}$, “Euler force” for $\underrightarrow{\alpha}^{ba/a} \times \underrightarrow{r}$, and “centrifugal force” for $\underrightarrow{\omega}^{ba} \times \left(\underrightarrow{\omega}^{ba} \times \underrightarrow{r}\right)$.

2 Inertial Measurement Unit

The IMU measures the angular velocity $\boldsymbol{\omega}_b^{ba}$, corrupted by a slow-changing bias $\mathbf{b}_b^g$ and noise $\mathbf{w}_b^g$,

$$\mathbf{u}^g = \boldsymbol{\omega}_b^{ba} + \mathbf{b}_b^g + \mathbf{w}_b^g \tag{1}$$

as well as specific force in the body frame, also corrupted by bias an noise,

$$\mathbf{u}^a = \mathbf{C}_{ab}^{\mathsf{T}}\left(\mathbf{a}_a^{ba/a/a} - \mathbf{g}_a\right) + \mathbf{b}_b^a + \mathbf{w}_b^a. \tag{2}$$

The IMU state is given by the attitude $\mathbf{C}_{ab}$, velocity $\mathbf{v}_a^{ba/a}$, and position $\mathbf{r}_a^{ba}$. The world frame is $\mathcal{F}_a$ and the body frame is $\mathcal{F}_b$. The kinematics of the platform are given by

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab}\boldsymbol{\omega}_b^{ba^{\times}}, \tag{3}$$

$$\dot{\mathbf{v}}_a^{ba/a} = \mathbf{a}_a^{ba/a/a}, \tag{4}$$

$$\dot{\mathbf{r}} = \mathbf{v}_a^{ba/a}. \tag{5}$$

The IMU kinematics are thus obtained by substituting the IMU measurements into the platform kinematics,

$$
\begin{aligned}
\dot{\mathbf{C}}_{ab} &= \mathbf{C}_{ab}\boldsymbol{\omega}_b^{ba^\times}, && (6)\\
&= \mathbf{C}_{ab}\left(\mathbf{u}^g - \mathbf{b}_b^g - \mathbf{w}_b^g\right), && (7)\\
\dot{\mathbf{v}}_a^{ba/a} &= \mathbf{a}_a^{ba/a/a} && (8)\\
&= \mathbf{C}_{ab}\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right) + \mathbf{g}_a. && (9)\\
\dot{\mathbf{r}} &= \mathbf{v}_a^{ba/a}, && (10)\\
\dot{\mathbf{b}}_b^g &= \mathbf{w}_b^{b_g}, && (11)\\
\dot{\mathbf{b}}_b^a &= \mathbf{w}_b^{b_a}. && (12)\\
& && (13)
\end{aligned}
$$

3 Earth Reference Frames

The inertial reference frame $\mathcal{F}_i$ does not rotate with the earth. The z-axis points toward the true north. Although reference frames have no physical location, it can be thought of as attached to Earth center but not rotating.

The ECEF frame of reference $\mathcal{F}_e$ has x basis vector pointing from the earth center toward the intersection of the equatorial plane and the prime meridian, the z basis pointing from earth center true north, and y completing the right-handed reference frame. Its angular velocity is given by $\underrightarrow{\omega}^{ei}$, physical vector with direction aligned with earth center to north pole, and magnitude equal to rate of rotation of the Earth. Getting the signs right requires considering the actual rotation,

$$\mathbf{C}_{ie_{k+1}} = \mathbf{C}_{ie_k} \operatorname{Exp}\left(\Delta t \boldsymbol{\omega}_e^{ei}\right). \tag{1}$$

Intuitively, Earth rotates counterclockwise. Therefore angle change for $\mathbf{C}_{ei}$ would be negative. Then, the fact that the DCM being considered is $\mathbf{C}_{ie}$ flips it back to positive again. Therefore,

$$\boldsymbol{\omega}_e^{ei^\mathsf{T}} = \begin{bmatrix} 0 & 0 & \left\|\boldsymbol{\omega}_e^{ei^\mathsf{T}}\right\| \end{bmatrix}^\mathsf{T}, \tag{2}$$

where $\left\|\boldsymbol{\omega}_e^{ei^\mathsf{T}}\right\|$ is magnitude of Earth rotation rate, about $360°/24$hr.

The other reference frame of interest is the East-North-Up frame $\mathcal{F}_{enu}$. This reference frame is constructed based on a specific location $\underrightarrow{p}$ at the Earth's surface, and as such may be written $\mathcal{F}_{enu}(\underrightarrow{p})$. The basis vectors point east, true north, and up, where up is specifically defined as *normal to the Earth surface.* This is important in discussion of geocentric and geodetic latitude.

The geocentric latitude is the angle between equatorial plane and the line from the the center of the earth and the surface point. The geodetic latitude is the angle between the equatorial plane and the *normal to the surface at the corresponding point*.

Geocentric latitude is geometrically intuitive, but rarely used. Geodetic latitude is typically used, and it is defined in such a way that position the surface of the ellipsoid is given by

$$\mathbf{r}_e^{pe} = \begin{bmatrix} (R_N(\phi) + h)\cos(\phi)\cos(\lambda) \\ (R_N(\phi) + h)\cos(\phi)\sin(\lambda) \\ (R_N(1 - e^2) + h)\sin(\phi) \end{bmatrix}. \tag{3}$$

4 North-East-Down Frame

The north-east-down frame is defined through the geodetic latitude convention, with the surface normal given point defining the down direction $\mathbf{d}_e$.

$$\mathbf{d}_e^{\mathsf{T}} = -\begin{bmatrix}\cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi\end{bmatrix}^{\mathsf{T}} \tag{1}$$

$$\mathbf{e}_e^{\mathsf{T}} = \begin{bmatrix}-\sin\lambda & \cos\lambda & 0\end{bmatrix}^{\mathsf{T}} \tag{2}$$

. The north vector resolved in ECEF frame is thus

$$\mathbf{n}_e = \mathbf{e}_e^{\times}\mathbf{d}_e = \begin{bmatrix}-\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi\end{bmatrix}. \tag{3}$$

The frame transformation from NED to ECEF frame is thus

$$\mathbf{C}_{ea} = \begin{bmatrix}\mathbf{n}_e & \mathbf{e}_e & \mathbf{d}_e\end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} -\sin\phi\cos\lambda & -\sin\lambda & -\cos\phi\cos\lambda \\ -\sin\phi\sin\lambda & \cos\lambda & -\cos\phi\sin\lambda \\ \cos\phi & 0 & -\sin\phi \end{bmatrix}. \tag{5}$$

4.1 Interlude: Active versus Passive Rotation

The commonly used direction cosine matrix is

$$\mathbf{v}_b = \mathbf{C}_{ba}\mathbf{v}_a. \tag{6}$$

This is what we commonly think of as "transformation from $\mathcal{F}_a$ to $\mathcal{F}_b$". This is a passive rotation. However, when we try to intuitively think of rotations, we think of a physical rotation. The two are inverses, as depicted in Fig. 1, which can cause confusion.

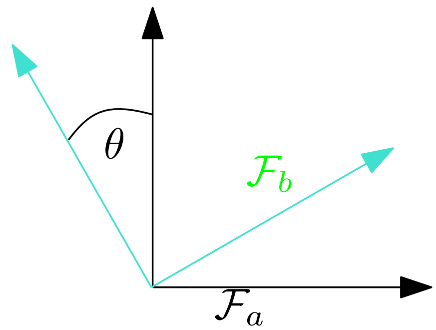
5 IMU Intrinsics

The gyroscope and accelerometer in an IMU may not be perfectly aligned. Let the accelerometer frame coincide with IMU frame, $\mathcal{F}_b$, and the gyroscope frame be denoted as $\mathcal{F}_g$. Misalignment and scale errors are present. Furthermore, acceleration may affect the gyroscope readings through the phenomenon of "g-sensitivity". Taking intrinsics into account, the true angular velocity and acceleration may be written as [1],

$$\boldsymbol{\omega}_b^{ba} = \mathbf{C}_{bg}\mathbf{D}^g\left(\mathbf{u}^g - \mathbf{T}^g\mathbf{a}_b^{ba/a/a} - \mathbf{b}^g - \mathbf{w}^g\right), \tag{1}$$

$$\mathbf{a}_b^{ba/a/a} - \mathbf{g}_b = \mathbf{D}^a\left(\mathbf{u}^a - \mathbf{b}^g - \mathbf{w}^g\right). \tag{2}$$

$$\mathbf{C}_{ba} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \mathbf{C}(-\theta)$$



Passive rotation is inverse of active rotation.

Figure 1: The "physically intuitive" rotation from $\mathcal{F}_a$ to $\mathcal{F}_b$, through a counterclockwise positive rotation angle, yields a rotation corresponding to $\mathrm{Exp}(-\theta)$.

The matrices $\mathbf{D}^g, \mathbf{D}^a$ are upper-triangular and parametrize scale and misalignment effects for the gyro and accelerometer, respectively. The matrix $\mathbf{T}^g$ parametrizes g-sensitivity. Different choices for the structure of $\mathbf{D}^g, \mathbf{D}^a, \mathbf{T}^g, \mathbf{C}_{bg}$, and choosing what to optimize correspond to the many IMU models analyzed in [1]. The main choices are: $\mathbf{T}^g$ fully populated or upper triangular, whether $\mathbf{C}_{bg}, \mathbf{D}^g$ are combined into a single fully populated matrix or not, and whether given intrinsics entries are held constant or not. Calibration of IMU intrinsics requires fully excited motion and an aiding sensor. The authors of [1] do not recommend online calibration with a camera due to parameter unobservabilities caused by degenerate motion.

6 Strapdown Navigation

Given an IMU, strapdown navigation takes into account rotation of the earth to obtain rotation and position estimates, of a vehicle with reference frame $\mathcal{F}_b$ and datum point b, with respect to the NED frame $\mathcal{F}_a$ and datum point a. and datum point. The estimated state is $\mathbf{C}_{ab}, \mathbf{v}_a^{ba}, \mathbf{r}_a^{ba}, \mathbf{b}_b^g, \mathbf{b}_b^a$. The attitude kinematics are given by

$$\begin{aligned} \dot{\mathbf{C}}_{ab} &= \mathbf{C}_{ab}\boldsymbol{\omega}_b^{ba^\times} && (1)\\ &= \mathbf{C}_{ab}\left(\boldsymbol{\omega}_b^{bi} + \boldsymbol{\omega}_b^{ia}\right)^\times && (2)\\ &= \mathbf{C}_{ab}\boldsymbol{\omega}_b^{bi^\times} + \mathbf{C}_{ab}\boldsymbol{\omega}_b^{ia^\times} && (3)\\ &= \mathbf{C}_{ab}\boldsymbol{\omega}_b^{bi^\times} + \mathbf{C}_{ab}\left(\mathbf{C}_{ab}^{\mathsf{T}}\boldsymbol{\omega}_a^{ia}\right)^\times. && (4) \end{aligned}$$

The quantity $\boldsymbol{\omega}_a^{ia}$ is manipulated as

$$\boldsymbol{\omega}_a^{ia} = \boldsymbol{\omega}_a^{ie} + \boldsymbol{\omega}_a^{ea} = \boldsymbol{\omega}_a^{ie}. \tag{5}$$

where, since NED frame does not rotate with respect to ECEF frame, $\boldsymbol{\omega}_a^{ea} = \mathbf{0}$. The attitude kinematics then become

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab}\boldsymbol{\omega}_b^{bi^\times} + \mathbf{C}_{ab}\left(\mathbf{C}_{ab}^\mathsf{T}\boldsymbol{\omega}_a^{ie}\right)^\times \tag{6}$$

$$= \mathbf{C}_{ab}\boldsymbol{\omega}_b^{bi^\times} + \boldsymbol{\omega}_a^{ie^\times}\mathbf{C}_{ab}, \tag{7}$$

where the identity $(\mathbf{Cu})^\times = \mathbf{Cu}^\times\mathbf{C}^\mathsf{T}$ was used. Expressing the kinematics in terms of measurements yields

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab}\left(\mathbf{u}^g - \mathbf{b}^g - \mathbf{w}_b^g\right)^\times + \boldsymbol{\omega}_a^{ie^\times}\mathbf{C}_{ab}, \tag{8}$$

with $\boldsymbol{\omega}_a^{ie}$ a constant that depends on the NED datum point, computed as

$$\boldsymbol{\omega}_a^{ie} = \mathbf{C}_{ae}\boldsymbol{\omega}_e^{ie} \tag{9}$$

$$= \begin{bmatrix} -\sin\phi\cos\lambda & -\sin\lambda & -\cos\phi\cos\lambda \\ -\sin\phi\sin\lambda & \cos\lambda & -\cos\phi\sin\lambda \\ \cos\phi & 0 & -\sin\phi \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 \\ 0 \\ \Omega_{\text{Earth}} \end{bmatrix} \tag{10}$$

$$= \Omega_{\text{Earth}} \begin{bmatrix} -\cos\phi \\ 0 \\ \sin\phi \end{bmatrix} \tag{11}$$

The velocity kinematics are derived by considering the acceleration transport theorem,

$$\underrightarrow{a}^{/a/a} = \underrightarrow{a}^{/b/b} + 2\underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{/b} + \underrightarrow{\alpha}^{ba/a} \times \underrightarrow{r} + \underrightarrow{\omega}^{ba} \times \left(\underrightarrow{\omega}^{ba} \times \underrightarrow{r}\right). \tag{12}$$

The accelerometer measures specific force, with acceleration being computed relative to the inertial frame

$$\underrightarrow{f} = \underrightarrow{a}^{ba/i/i} - \underrightarrow{g}, \tag{13}$$

where $\underrightarrow{a}^{ba/i/i}$ is computed using angular transport theorem, with the replacements $\mathcal{F}_a \to \mathcal{F}_i$ and $\mathcal{F}_b \to \mathcal{F}_a$,

$$\underrightarrow{a}^{ba/i/i} = \underrightarrow{a}^{ba/a/a} + 2\underrightarrow{\omega}^{ai} \times \underrightarrow{v}^{/a} + \underrightarrow{\alpha}^{ai/i} \times \underrightarrow{r}^{ba} + \underrightarrow{\omega}^{ai} \times \left(\underrightarrow{\omega}^{ai} \times \underrightarrow{r}^{ba}\right). \tag{14}$$

The angular acceleration $\underrightarrow{\alpha}^{ai/i} = \underrightarrow{\alpha}^{ei/i}$ is rate of change of angular velocity of the Earth, and is thus zero. The $\underrightarrow{\omega}^{ai} \times \left(\underrightarrow{\omega}^{ai} \times \underrightarrow{r}^{ba}\right)$ term is on the order of Ω_{Earth}^2, and since the local NED position $\underrightarrow{r}^{ba}$ is fairly small, the whole term is order of Ω_{Earth}^2. Therefore,

$$\underrightarrow{a}^{ba/i/i} = \underrightarrow{a}^{ba/a/a} + 2\underrightarrow{\omega}^{ai} \times \underrightarrow{v}^{ba/a} + \underrightarrow{\alpha}^{ai/i} \times \underrightarrow{r}^{ba} + \underrightarrow{\omega}^{ai} \times \left(\underrightarrow{\omega}^{ai} \times \underrightarrow{r}^{ba}\right) \tag{15}$$

$$= \underrightarrow{a}^{ba/a/a} + 2\underrightarrow{\omega}^{ai} \times \underrightarrow{v}^{ba/a} + \underrightarrow{\omega}^{ai} \times \left(\underrightarrow{\omega}^{ai} \times \underrightarrow{r}^{ba}\right) \tag{16}$$

$$\approx \underrightarrow{a}^{ba/a/a} + 2\underrightarrow{\omega}^{ai} \times \underrightarrow{v}^{ba/a}. \tag{17}$$

Therefore, the inertial-frame specific force expression (13) may be written in terms of local acceleration as

$$\underset{\rightarrow}{f} = \underset{\rightarrow}{a}^{ba/a/a} + 2\,\underset{\rightarrow}{\omega}^{ai} \times \underset{\rightarrow}{v}^{ba/a} - \underset{\rightarrow}{g}. \tag{18}$$

The accelerometer measures specific force in the body frame, subject to bias and noise terms,

$$\mathbf{u}^a = \mathbf{f}_b + \mathbf{b}_b^a + \mathbf{w}_b^a, \tag{19}$$

meaning that (18) may be resolved in the body frame and manipulated as

$$\begin{aligned} \mathbf{f}_b &= \mathbf{u}^a - \mathbf{b}_b - \mathbf{w}_b^a & (20)\\ &= \mathbf{a}_b^{ba/a/a} + 2\boldsymbol{\omega}_b^{ai\times}\mathbf{v}_b^{ba/a} - \mathbf{g}_b & (21)\\ &= \mathbf{a}_b^{ba/a/a} + 2\boldsymbol{\omega}_b^{ei\times}\mathbf{v}_b^{ba/a} - \mathbf{g}_b. & (22) \end{aligned}$$

Where last line is obtained by substituting (5) into the Coriolis term. Its important to keep in mind we are still overloading b, a to denote both reference frames and the datum points, so it is important to be mindful of which side of the slash they come up in. Furthermore, superscripts of angular velocity $\boldsymbol{\omega}_b^{ba}$ denote reference frames as well.

Isolating for the acceleration and resolving in NED frame yields

$$\mathbf{a}_a^{ba/a/a} = \mathbf{C}_{ab}\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_a^a - 2\boldsymbol{\omega}_b^{ei\times}\mathbf{v}_b^{ba/a}\right) + \mathbf{g}_a. \tag{23}$$

Furthermore, $\boldsymbol{\omega}_b^{ei\times} = \left(\mathbf{C}_{ab}^{\mathsf{T}}\boldsymbol{\omega}_a^{ei}\right)^{\times} = \mathbf{C}_{ab}^{\mathsf{T}}\boldsymbol{\omega}_a^{ei^{\times}}\mathbf{C}_{ab}$, yielding

$$\begin{aligned} \mathbf{a}_a^{ba/a/a} &= \mathbf{C}_{ab}\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a - 2\boldsymbol{\omega}_b^{ei\times}\mathbf{v}_b^{ba/a}\right) + \mathbf{g}_a. & (24)\\ &= \mathbf{C}_{ab}\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right) - 2\boldsymbol{\omega}_a^{ei\times}\mathbf{C}_{ab}\mathbf{v}_b^{ba/a} + \mathbf{g}_a & (25)\\ &= \mathbf{C}_{ab}\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right) + 2\boldsymbol{\omega}_a^{ie\times}\mathbf{v}_a^{ba/a} + \mathbf{g}_a. & (26) \end{aligned}$$

Therefore, IMU kinematics for strapdown navigation that take into account the rotating Earth effect are given by

$$\begin{aligned} \dot{\mathbf{C}}_{ab} &= \mathbf{C}_{ab}\left(\mathbf{u}^g - \mathbf{b}_b^g - \mathbf{w}_b^g\right)^{\times} + \boldsymbol{\omega}_a^{ie\times}\mathbf{C}_{ab}, & (27)\\ \dot{\mathbf{v}}_a^{ba/a} &= \mathbf{C}_{ab}\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right) + 2\boldsymbol{\omega}_a^{ie\times}\mathbf{v}_a^{ba/a} + \mathbf{g}_a, & (28)\\ \dot{\mathbf{r}}_a^{ba} &= \mathbf{v}_a^{ba/a}, & (29)\\ \dot{\mathbf{b}}_b^g &= \mathbf{w}^g, & (30)\\ \dot{\mathbf{b}}_b^a &= \mathbf{w}^a. & (31) \end{aligned}$$

7 Leveling and Gyrocompassing

For a static IMU, (27) and (28) become

$$\begin{aligned} \mathbf{C}_{ab}\left(\mathbf{u}^g - \mathbf{b}_b^g - \mathbf{w}_b^g\right)^{\times} + \boldsymbol{\omega}_a^{ie\times}\mathbf{C}_{ab} = \mathbf{0}, & \quad (1)\\ \mathbf{C}_{ab}\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_a^a\right) + \mathbf{g}_a = \mathbf{0}. & \quad (2) \end{aligned}$$

where rotation equation is manipulated as

$$(\mathbf{u}^g - \mathbf{b}_b^g - \mathbf{w}_b^g)^\times + \mathbf{C}_{ab}^\mathsf{T} \boldsymbol{\omega}_a^{ie\times} \mathbf{C}_{ab} = \mathbf{0}, \tag{3}$$

$$(\mathbf{u}^g - \mathbf{b}_b^g - \mathbf{w}_b^g)^\times + \left(\mathbf{C}_{ab}^\mathsf{T} \boldsymbol{\omega}_a^{ie}\right)^\times = \mathbf{0}, \tag{4}$$

$$(\mathbf{u}^g - \mathbf{b}_b^g - \mathbf{w}_b^g) + \mathbf{C}_{ab}^\mathsf{T} \boldsymbol{\omega}_a^{ie} = \mathbf{0}. \tag{5}$$

Putting together the static gyro and accelerometer equations yields

$$\mathbf{C}_{ab} \left(\mathbf{u}^g - \mathbf{b}_b^g - \mathbf{w}_b^g\right) + \boldsymbol{\omega}_a^{ie} = \mathbf{0}, \tag{6}$$

$$\mathbf{C}_{ab} \left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_a^a\right) + \mathbf{g}_a = \mathbf{0}. \tag{7}$$

For a known bias, and noise averaged out, this directly corresponds to Wahba's problem,

$$\mathbf{C}_{ab} = \arg\min_{\mathbf{C}} \sum_{i=1}^{2} \alpha_i \left\| \mathbf{w}_i - \mathbf{C}\mathbf{v}_i \right\|. \tag{8}$$

This corresponds to coarse alignment. The gyro bias error magnitude is comparable to the Earth rotation rate. In the static case, $\mathbf{C}_{ab} = \text{const}$, if biases are included in the state, the biases are not observable. The problem is therefore solved under assumption of known biases for coarse alignment. For gyro bias in particular, the bias error is high compared to magnitude of Earth rotation rate. Fine alignment is thus used to refine bias estimates.

Fine alignment requires running Kalman filter type of approaches. Exteroceptive measurements such as Zero Velocity (ZUPT), Zero Angular Velocity (ZARU), GNSS are fused to refine bias estimates throughout operation. Motion is required for full observability, depending on the type of measurements used. A section of literature is devoted to this. It is on my TODO list to review it and add it here.

8 Preintegration

This section is a work in progress, updated as I update my understanding.

The preintegration section is largely based off of Mitchell Cohen's document (https://mitchellcohen3.github.io/assets/pdf/notes/imu_kinematics.pdf).

9 Analytic Integration of IMU Kinematics

For the attitude component, for a constant $\boldsymbol{\omega}_b^{ba}$ over a given time interval Δt, the *exact* solution is given by

$$\mathbf{C}_{ab}(t + \Delta t) = \mathbf{C}_{ab}(t) \operatorname{Exp}(\boldsymbol{\omega}_b^{ba} \Delta t), \tag{1}$$

which can be chained for piecewise constant $\boldsymbol{\omega}_{b_1}^{ba}, \ldots, \boldsymbol{\omega}_{b_k}^{ba}$ as

$$\mathbf{C}_{ab}(t + \Delta t_1 + \cdots + \Delta t_k) = \mathbf{C}_{ab}(t) \operatorname{Exp}(\boldsymbol{\omega}_{b_1}^{ba} \Delta t_1) \ldots \operatorname{Exp}(\boldsymbol{\omega}_{b_k}^{ba} \Delta t_k). \tag{2}$$

For small Δt, $\operatorname{Exp}(\boldsymbol{\omega}_{b_k}^{ba}\Delta t_k) \approx \mathbf{1} + (\boldsymbol{\omega}_{b_k}^{ba}\Delta t_k)^\times$ and thus $\operatorname{Exp}(\boldsymbol{\omega}_{b_1}^{ba}\Delta t_1)\operatorname{Exp}(\boldsymbol{\omega}_{b_2}^{ba}\Delta t_2) = \operatorname{Exp}(\boldsymbol{\omega}_{b_1}^{ba}\Delta t_1 + \boldsymbol{\omega}_{b_2}^{ba}\Delta t_2)$. Integration is thus allowed as

$$\begin{aligned}
\mathbf{C}_{ab}(t) &= \mathbf{C}_{ab}(0)\operatorname{Exp}\left(\int_0^t \boldsymbol{\omega}_b^{ba}\mathrm{d}s\right) && (3)\\
&= \mathbf{C}_{ab}(0)\Delta\mathbf{C}_t, && (4)
\end{aligned}$$

where $\Delta\mathbf{C}_t = \operatorname{Exp}\left(\int_0^t \boldsymbol{\omega}_b^{ba}\mathrm{d}s\right)$ is the attitude relative motion increment (RMI). The velocity integration follows as

$$\begin{aligned}
\mathbf{v}_a^{ba/a}(t) &= \mathbf{v}_0 + \int_0^t \mathbf{a}_a^{ba/a/a}\mathrm{d}s && (5)\\
&= \mathbf{v}_0 + \int_0^t \mathbf{a}_a^{ba/a/a}\mathrm{d}s && (6)\\
&= \mathbf{v}_0 + \int_0^t \left(\mathbf{C}_{ab}\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right) + \mathbf{g}_a\right)\mathrm{d}s && (7)\\
&= \mathbf{v}_0 + \mathbf{g}_a t + \mathbf{C}_0\int_0^t \Delta\mathbf{C}_s\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right)\mathrm{d}s && (8)\\
&= \mathbf{v}_0 + \mathbf{g}_a t + \mathbf{C}_0\Delta\mathbf{v}_t, && (9)
\end{aligned}$$

where $\Delta\mathbf{v}_t = \int_0^t \Delta\mathbf{C}_s\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right)\mathrm{d}s$ is the velocity RMI. The position is given by

$$\begin{aligned}
\mathbf{r} &= \mathbf{r}_0 + \int_0^t \mathbf{v}_a^{ba/a}\mathrm{d}u && (10)\\
&= \mathbf{r}_0 + \int_0^t \mathbf{v}_0 + \mathbf{g}_a t + \mathbf{C}_0\Delta\mathbf{v}_u\mathrm{d}u && (11)\\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}_a t^2 + \mathbf{C}_0\int_0^t \Delta\mathbf{v}_u\mathrm{d}u && (12)\\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}_a t^2 + \mathbf{C}_0\int_0^t\int_0^u \Delta\mathbf{C}_s\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right)\mathrm{d}s\mathrm{d}u && (13)\\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}_a t^2 + \mathbf{C}_0\Delta\mathbf{r}_t && (14)
\end{aligned}$$

where $\Delta\mathbf{r}_t = \int_0^t\int_0^u \Delta\mathbf{C}_s\left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a\right)\mathrm{d}s\mathrm{d}u$ is the position RMI. Therefore, the preintegrated IMU kinematics are given by

$$\begin{aligned}
\mathbf{C}_{ab}(t) &= \mathbf{C}_{ab}(0)\Delta\mathbf{C}_t, && (15)\\
\mathbf{v}_a^{ba/a/a} &= \mathbf{v}_0 + \mathbf{g}_a t + \mathbf{C}_0\Delta\mathbf{v}_t, && (16)\\
\mathbf{r}_a^{ba} &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}_a t^2 + \mathbf{C}_0\Delta\mathbf{r}_t, && (17)
\end{aligned}$$

where

$$\Delta \mathbf{C}_t = \operatorname{Exp}\left(\int_0^t \boldsymbol{\omega}_b^{ba} \mathrm{d} \right), \tag{18}$$

$$\Delta \mathbf{v}_t = \int_0^t \Delta \mathbf{C}_s \left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a \right) \mathrm{d}s, \tag{19}$$

$$\Delta \mathbf{r}_t = \int_0^t \int_0^u \Delta \mathbf{C}_s \left(\mathbf{u}^a - \mathbf{b}_b^a - \mathbf{w}_b^a \right) \mathrm{d}s \mathrm{d}u. \tag{20}$$

The RMI depends on the initial pose, the bias, and the noise only.

10 $SE_2(3)$ Formulation

The preintegrated kinematics may be written in matrix form as

$$\begin{aligned}
\begin{bmatrix} \mathbf{C}_{ab} & \mathbf{v}_a^{ba} & \mathbf{r}_a^{ba} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_0 & \mathbf{v}_0 + \mathbf{g}_a t & \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}_a t^2 \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{C}_t & \Delta\mathbf{v}_t & \Delta\mathbf{r}_t \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} && (1) \\
&= \begin{bmatrix} \mathbf{1} & \mathbf{g}_a t & \frac{1}{2}\mathbf{g}_a t^2 \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{C}_0 & \mathbf{v}_0 & \mathbf{r}_0 + \mathbf{v}_0 t \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{C}_t & \Delta\mathbf{v}_t & \Delta\mathbf{r}_t \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} && (2) \\
&= \boldsymbol{\Gamma}_t \boldsymbol{\Phi}_t(\mathbf{T}_0) \boldsymbol{\Upsilon}_t. && (3)
\end{aligned}$$

11 Mean Propagation

The RMIs involve analytic integrals that must be evaluated using specific assumptions. Commonly used are constant acceleration and constant IMU measurement.

12 Preintegration

Given two poses $\mathbf{T}_i, \mathbf{T}_j$, they are related through the IMU kinematics as

$$\mathbf{T}_j = \boldsymbol{\Gamma}_{ij} \boldsymbol{\Phi}(\mathbf{T}_i) \boldsymbol{\Upsilon}_{ij}, \tag{1}$$

such that the RMI may be isolated as

$$\boldsymbol{\Upsilon}_{ij} = \left(\boldsymbol{\Gamma}_{ij} \boldsymbol{\Phi}(\mathbf{T}_i) \right)^{-1} \mathbf{T}_j. \tag{2}$$

The RMI $\boldsymbol{\Upsilon}_{ij}$ depends only on the measurements $\mathbf{u}^a, \mathbf{u}^g$ as well as the biases $\mathbf{b}^a, \mathbf{b}^g$.

The notation changes slightly from here on out. There are two RMIs. The predicted RMI from measurements is denoted $\hat{\boldsymbol{\Upsilon}}_{ij}$ while the one computed from the states is denoted $\boldsymbol{\Upsilon}_{ij}$ and may be expanded as

$$
\begin{aligned}
\boldsymbol{\Upsilon}_{ij} &= \left(\boldsymbol{\Gamma}_{ij}\boldsymbol{\Phi}(\mathbf{T}_i)\right)^{-1}\mathbf{T}_j & (3)\\
&= \begin{bmatrix} \mathbf{C}_i & \mathbf{v}_i + \mathbf{g}_a\Delta t_{ij} & \mathbf{r}_i + \mathbf{v}_i\Delta t_{ij} + \frac{1}{2}\mathbf{g}_a\Delta t_{ij}^2 \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C}_j & \mathbf{v}_j & \mathbf{r}_j \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} & (4)\\
&= \begin{bmatrix} \mathbf{C}_i^{\mathsf{T}} & -\mathbf{C}_i^{\mathsf{T}}\left(\mathbf{v}_i + \mathbf{g}_a\Delta t_{ij}\right) & -\mathbf{C}_i^{\mathsf{T}}\left(\mathbf{r}_i + \mathbf{v}_i\Delta t_{ij} + \frac{1}{2}\mathbf{g}_a\Delta t_{ij}^2\right) \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{C}_j & \mathbf{v}_j & \mathbf{r}_j \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} & (5)\\
&= \begin{bmatrix} \mathbf{C}_i^{\mathsf{T}}\mathbf{C}_j & \mathbf{C}_i^{\mathsf{T}}\left(\mathbf{v}_j - \mathbf{v}_i - \Delta t_{ij}\mathbf{g}_a\right) & \mathbf{C}_i^{\mathsf{T}}\left(\mathbf{r}_j - \mathbf{r}_i - \mathbf{v}_i\Delta t_{ij} - \frac{1}{2}\mathbf{g}_a\Delta t_{ij}^2\right) \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} & (6)\\
&= \begin{bmatrix} \Delta\mathbf{C}_{ij} & \Delta\mathbf{v}_{ij} & \Delta\mathbf{r}_{ij} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix}. & (7)
\end{aligned}
$$

13 GNSS

This section is a work in progress, updated as I update my understanding. A measurement to a GNSS satellite consists of a pseudorange and a carrier phase. Doppler measurements are also present. The satellite broadcasts a radio wave, that can be visualized as a stationary wave, such that the robot is located at a specific location on this wave.

Denote $t_{\text{event}}^{\text{clock}}$ denote the time of occurence of an event, measured using a given clock. The time of the master clock is denoted t^{master}.

For a satellite s and a receiver b, the time of transmission in the satellite clock is recorded in the signal. Once it reaches the receiver, the receiver computes the time difference

$$
\begin{aligned}
\Delta t &= t_{\text{rec.}}^{b} - t_{\text{trans.}}^{s} & (1)\\
&= t_{\text{rec.}}^{\text{master}} + \delta t_b - t_{\text{trans.}}^{s} - \delta t_s & (2)\\
&= \Delta t_{\text{true}} + \delta t_b - \delta t_s. & (3)
\end{aligned}
$$

The quantities δt_b, δt_s denote the time offsets of the receiver and satellite clocks with respect to the master clock.

Let the position of the satellite be denoted as the physical vector $\underrightarrow{s}$. Let the position of the receiver be denoted as the physical vector $\underrightarrow{b}$. Then, the pseudorange measurement is obtained by premultiplying (3) by the speed of light and adding errors due to ionosphere, troposphere, and white noise effects,

$$
\rho = c\Delta t = \left\| \underrightarrow{s} - \underrightarrow{b} \right\| + c\left(\delta t_b - \delta t_s\right) - I + T + w^{\rho}. \quad (4)
$$

There is the question of how $t_{\text{trans.}}^{s}$ is determined by the receiver. The signal emitted by the satellite carries a code, sequence of zeros and ones, which is repeated. The receiver has a replica of this code, and aligns the two. The pseudorange obtained through this method is limited in accuracy, based on the code resolution. This accuracy is on the order of meters.

Furthermore, there is a carrier phase measurement. The receiver measures the phase of the carrier signal very accurately. Denote ϕ as the amount of cycles spanning the distance between receiver and the satellite, and λ be the signal wavelength. Wavelength is on the order of tens of cm. The carrier phase equation is written

$$\lambda\phi = \left\| \underrightarrow{s} - \underrightarrow{b} \right\| + c\left(\delta t_b - \delta t_s\right) + \lambda N - I + T + w^{\phi}, \tag{5}$$

where N is the number of cycles between the receiver and the satellite *at the time of lock*, and ϕ is the measured phase. At the time of lock, $\phi(t_0) \in [0, 1)$, but can go outside of these bounds as time progresses, until it loses lock. The integer N is unknown. It is constrained by the pseudorange measurement to a given interval, but must be solved for.

13.1 Real Time Kinematics (RTK)

RTK is a GNSS navigation approach that assumes two receivers. In practice, this requires a basestation.

13.1.1 Differencing

Single differencing relies on having two receivers. In practice, this means adding a base station. The differencing then consists of writing the pseudorange equations for the two receivers,

$$\rho_1 = c\Delta t = \left\| \underrightarrow{s} - \underrightarrow{b_1} \right\| + c\left(\delta t_{b_1} - \delta t_s\right) - I + T + w_1^{\rho}, \tag{6}$$

$$\rho_2 = c\Delta t = \left\| \underrightarrow{s} - \underrightarrow{b_2} \right\| + c\left(\delta t_{b_2} - \delta t_s\right) - I + T + w_1^{\rho}, \tag{7}$$

and taking the difference to yield

$$\Delta\rho_{1,2} = \left\| \underrightarrow{s} - \underrightarrow{b_1} \right\| - \left\| \underrightarrow{s} - \underrightarrow{b_2} \right\| + c\left(\delta t_{b_1} - \delta t_{b_2}\right) + \Delta I_{1,2} + \Delta T_{1,2} w^{\Delta\rho}. \tag{8}$$

For differenced quantities such as ΔI, $\Delta I_{i,j}$ denotes the ionosphere difference for the two receivers, $I_i - I_j$.

If the base and receiver are close enough, the ionosphere and troposphere differences are approximately zero. Writing the single difference equation for two satellites yields

$$\Delta\rho_{1,2}^1 = \left\| \underrightarrow{s_1} - \underrightarrow{b_1} \right\| - \left\| \underrightarrow{s_1} - \underrightarrow{b_2} \right\| + c\left(\delta t_{b_1} - \delta t_{b_2}\right) + \Delta I_{1,2}^2 + \Delta T_{1,2}^2 w^{\Delta\rho}, \tag{9}$$

$$\Delta\rho_{1,2}^2 = \left\| \underrightarrow{s_2} - \underrightarrow{b_1} \right\| - \left\| \underrightarrow{s_2} - \underrightarrow{b_2} \right\| + c\left(\delta t_{b_1} - \delta t_{b_2}\right) + \Delta I_{1,2}^2 + \Delta T_{1,2}^2 w^{\Delta\rho}. \tag{10}$$

The double difference is obtained as

$$\Delta\rho_{1,2}^{1,2} = \left\| \underrightarrow{s_1} - \underrightarrow{b_1} \right\| - \left\| \underrightarrow{s_1} - \underrightarrow{b_2} \right\| - \left\| \underrightarrow{s_2} - \underrightarrow{b_1} \right\| + \left\| \underrightarrow{s_2} - \underrightarrow{b_2} \right\| + \Delta I_{1,2}^{1,2} + \Delta T_{1,2}^{1,2} w^{\Delta\rho}. \tag{11}$$

Similarly, the carrier phase double difference is written as follows. The phase ϕ_i^j is the phase measured by the i'th receiver, receiving signal from j'th satellite.

$$
\begin{aligned}
\lambda\Delta\phi_{1,2}^{1,2} &= \Delta\phi_{1,2}^{1} - \Delta\phi_{1,2}^{2} && (12)\\
&= \left(\Delta\phi_1^1 - \Delta\phi_1^2\right) - \left(\Delta\phi_2^1 - \Delta\phi_2^2\right) && (13)\\
&= \left\|\underrightarrow{s_1} - \underrightarrow{b_1}\right\| - \left\|\underrightarrow{s_1} - \underrightarrow{b_2}\right\| - \left\|\underrightarrow{s_2} - \underrightarrow{b_1}\right\| + \left\|\underrightarrow{s_2} - \underrightarrow{b_2}\right\| + \lambda N_{1,2}^{1,2} + \Delta I_{1,2}^{1,2} + \Delta T_{1,2}^{1,2} w^{\Delta\rho}, && (14)
\end{aligned}
$$

where the double difference integer ambiguity is given by

$$
N_{1,2}^{1,2} = \left(\left(N_1^1 - N_1^2\right) - \left(N_2^1 - N_2^2\right)\right). \tag{15}
$$

Summary

The double differenced equations for pseudorange (11)**, and** (14) **may be used as measurements in an EKF. The receiver clock states are removed, but base station is required. Integer ambiguity needs to be solved for.**

13.1.2 EKF for Differenced Formulation

The pseudorange and carrier phase measurements may be used in an EKF. All positions are in ECEF frame of reference $\mathcal{F}_e$. Given M satellites, there are $M-1$ double-differenced measurements, indexed m, each with their own integer ambiguity N_m. The integer ambiguities form a vector $\mathbf{N} \in \mathbb{Z}^{m-1}$. The state consists of the receiver position in the ECEF frame as well as the integer resolution state, $\mathcal{X} = \left(\mathbf{r}_e^{b(t)e}, \mathbf{N}\right) \in \mathbb{R}^3 \times \mathbb{R}$, where $\mathbf{r}_e^{b(t)e}$ is a time-dependent quantity while $\mathbf{N}$ is time-independent. The quantity N is solved for as a float.

13.1.3 Least-Squares Ambiguity Decorrelation Adjustement

The EKF provides an estimate for $\mathbf{N}$, with a Gaussian distribution $\mathbf{N} \sim \mathcal{N}(\check{\mathbf{N}}, \mathbf{P})$. However, $\check{\mathbf{N}}$ is a float, whereas the actual value is an integer, $\mathbf{N} \in \mathbb{Z}^{m-1}$. Simple rounding does not work, because rounding does not take into account the correlation between the entries of $\mathbf{N}$. Instead, the Mahalanobis distance is minimized over integers,

$$
\hat{\mathbf{N}} = \underset{\mathbf{N}\in\mathbb{Z}^{m-1}}{\arg\min} \left(\mathbf{N} - \check{\mathbf{N}}\right)^{\mathsf{T}} \mathbf{P}^{-1} \left(\mathbf{N} - \check{\mathbf{N}}\right). \tag{16}
$$

The LAMBDA algorithm itself does some clever things to make this fast and work well. Once LAMBDA runs, it will provide two top candidates. If the first one is significantly better, passes a ratio test (for the cost), then $\mathbf{N}$ is fixed as a constraint. If not, keep refining $\mathbf{N}$ as a float.

13.2 Interlude: Doppler Shift

Given a car moving away from a person with a speed v_s, emitting a sound at frequency f_s, the person will hear a shifted frequency f_r. Between each wave cycle, the car will move a distance Δx, causing the receiver to perceive a wavelength $\lambda_r = \lambda_s + \Delta x$. Formally, the effect is derived as

$$
\lambda_r = \lambda_s + v_s \Delta t, \tag{17}
$$

where Δt is time between cycles, $\Delta t = 1/f_s$,

$$\lambda_r = \lambda_s + \frac{v_s}{f_s}. \tag{18}$$

Dividing through by speed of propagation through medium v_m and bringing to common denominator yields

$$\begin{aligned} \frac{\lambda_r}{v_m} &= \frac{\lambda_s}{v_m} + \frac{v_s}{v_m f_s} && (19) \\ &= \frac{f_s \lambda_s + v_s}{v_m f_s} && (20) \\ &= \frac{v_m + v_s}{v_m f_s}, && (21) \\ & && (22) \end{aligned}$$

where constant wave speed in medium equation was used, $v_m = \lambda f$, true for both source and receiver. Keeping in mind $v_m = f\lambda$ allows to solve for frequency as

$$\begin{aligned} \frac{v_m}{\lambda_r} &= \frac{v_m f_s}{v_m + v_s}, && (23) \\ f_r &= f_s \frac{v_m}{v_m + v_s}. && (24) \end{aligned}$$

Given a common inertial frame $\mathcal{F}_a$, a source reference point s, and a receiver reference point r, the relative velocity projected onto the normal between source and receiver is used,

$$v_s = \underset{\rightarrow}{v}^{sr/a/a} \cdot \underset{\rightarrow}{r}^{sr}. \tag{25}$$

In practice, measured Doppler shift is $f_r - f_s$, which may be written

$$\begin{aligned} f_r - f_s &= f_s \frac{v_m}{v_m + v_s} - f_s && (26) \\ &= f_s \left(\frac{v_m}{v_m + v_s} - 1 \right) && (27) \\ &= \frac{-v_s}{v_m + v_s} f_s. && (28) \end{aligned}$$

For v_s/v_m small,

$$\begin{aligned} f_r - f_s &= \frac{v_s}{v_m} \frac{-1}{1 + v_s/v_m} f_s && (29) \\ &\approx -\frac{v_s}{v_m} f_s && (30) \\ &= -\frac{v_s}{v_m} f_s && (31) \\ &= -\frac{v_s}{\lambda}. && (32) \end{aligned}$$

Doppler shift measurement may then be written, similar to pseudorange

$$d = \underset{\rightarrow}{v}^{sr/a/a} \cdot \underset{\rightarrow}{r}^{sr} + c \left(\delta \dot{t}_b - \delta \dot{t}_s \right) + w^d. \tag{33}$$

References

[1] Y. Yang, P. Geneva, X. Zuo, and G. Huang, “Online self-calibration for visual-inertial navigation: Models, analysis, and degeneracy,” *IEEE Transactions on Robotics*, vol. 39, no. 5, pp. 3479–3498, 2023.