Semidefinite Relaxations of Quadratically Constrained Quadratic Programs

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This document provides a derivation of the semidefinite relaxation for quadratically constrainted quadratic (QCQP) programs and shows it is the bidual (dual of the dual) of the QCQP.

1 Quadratically Constrained Quadratic Programs

A homogeneous QCQP is given by

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} \\ \text{subject to} \quad & \mathbf{x}^\mathsf{T}\mathbf{A}_i\mathbf{x} + b_i = 0, \quad i = 1, \dots, m \end{aligned} \tag{QCQP}$$

The Lagrangian is given by

$$L(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} + \sum_{i=1}^{m} \lambda_i (b_i + \mathbf{x}^\mathsf{T}\mathbf{A}_i\mathbf{x}) \tag{1}$$

$$= \boldsymbol{\lambda}^\mathsf{T}\mathbf{b} + \mathbf{x}^\mathsf{T}\left(\mathbf{Q} + \sum_{i=1}^{m} \lambda_i \mathbf{A}_i\right)\mathbf{x}, \tag{2}$$

such that the dual problem is given by

$$\begin{aligned} \underset{\boldsymbol{\lambda}}{\text{maximize}} \quad & \boldsymbol{\lambda}^\mathsf{T}\mathbf{b} \\ \text{subject to} \quad & \mathbf{Q} + \sum_{i=1}^{m} \lambda_i \mathbf{A}_i \succcurlyeq 0 \end{aligned} \tag{Dual}$$

Notice that the PSD constraint is given by half of the Hessian of the Lagrangian (2) with respect to **x**, which is denoted

$$\mathcal{H}(\boldsymbol{\lambda}) = \frac{1}{2}\frac{\partial^2 L(\boldsymbol{\lambda}, \mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}^\mathsf{T}} = \mathbf{Q} + \sum_{i=1}^{m} \lambda_i \mathbf{A}_i. \tag{3}$$

To derive the bidual, the Lagrangian of the dual problem is given by

$$L(\boldsymbol{\lambda}, \mathbf{S}) = \boldsymbol{\lambda}^{\mathsf{T}}\mathbf{b} + \left\langle \mathbf{Q} + \sum_{i=1}^{m} \lambda_i \mathbf{A}_i, \mathbf{S} \right\rangle \tag{4}$$

$$= \boldsymbol{\lambda}^{\mathsf{T}}\mathbf{b} + \operatorname{tr}\left(\left(\mathbf{Q} + \sum_{i=1}^{m} \lambda_i \mathbf{A}_i\right)\mathbf{S}\right) \tag{5}$$

$$= \sum_{i=1}^{m} \lambda_i b_i + \operatorname{tr}(\mathbf{QS}) + \sum_{i=1}^{m} \lambda_i \operatorname{tr}(\mathbf{A}_i\mathbf{S}) \tag{6}$$

$$= \operatorname{tr}(\mathbf{QS}) + \sum_{i=1}^{m} \lambda_i (b_i + \operatorname{tr}(\mathbf{A}_i\mathbf{S})), \tag{7}$$

where $\mathbf{S} \succcurlyeq 0$ is the bidual Lagrange multiplier and $\boldsymbol{\lambda}$ are the Lagrange multipliers of the original dual. The inner product notation is used as $\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{M} \sum_{j=1}^{N} \mathbf{A}_{ij}\mathbf{B}_{ij} = \operatorname{tr}(\mathbf{AB}^{\mathsf{T}})$. Since this is the Lagrangian of a maximization problem, the constraint cost terms are added instead of subtracted. The bidual dual function is given by

$$g(\mathbf{S}) = \sup_{\boldsymbol{\lambda}} \left(\operatorname{tr}(\mathbf{QS}) + \sum_{i=1}^{m} \lambda_i (b_i + \operatorname{tr}(\mathbf{A}_i\mathbf{S})) \right), \tag{8}$$

which means that $b_i + \operatorname{tr}(\mathbf{A}_i\mathbf{S}) = 0$ for feasibility of the bidual optimization problem, which is given by

$$\begin{aligned} \underset{\mathbf{S}}{\text{minimize}} \quad & \langle \mathbf{Q}, \mathbf{S} \rangle \\ \text{subject to} \quad & \langle \mathbf{A}_i, \mathbf{S} \rangle + b_i = 0. \end{aligned} \tag{Bidual-SDP}$$

The problems (Bidual-SDP) and (Dual) are the duals of each other. Furthermore, (Dual) is the dual of the non-convex QCQP (QCQP). The optimal values of the three problems are related by val(Dual) $\leq$ val(Bidual-SDP) $\leq$ val(QCQP).

Furthermore, (Bidual-SDP) is a rank relaxation of (QCQP). Since $\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} = \operatorname{tr}(\mathbf{x}\mathbf{x}^{\mathsf{T}}\mathbf{Q})$, letting $\mathbf{Z} = \mathbf{x}\mathbf{x}^{\mathsf{T}}$ yields an equivalent problem to (QCQP) as

$$\begin{aligned} \underset{\mathbf{Z}}{\text{minimize}} \quad & \langle \mathbf{Z}, \mathbf{Q} \rangle \\ \text{subject to} \quad & \langle \mathbf{Z}, \mathbf{Q} \rangle + b_i = 0, \quad i = 1, \ldots, m \\ & \operatorname{rank}(\mathbf{Z}) = 1. \end{aligned}$$

Relaxing the rank constraint into a positive semidefinite constraint yields exactly (Bidual-SDP).